Ballistic Walking Design via Impulsive Control

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Abstract: Mathematical models of a biped and quadruped walking are considered. The planar five-link biped consists of a one-link trunk and two identical two-link legs. In the single support motion (swing phase), the biped has five degrees of freedom and is described by a system of nonlinear ordinary differential equations of the 10th order. These equations are written in a visible matrix form. A seven-link planar biped with massless feet is also considered. In this paper, the swing motion of the biped is assumed a ballistic (passive) one. There are no active torques in the interlink joints during the single support motion—only the gravity force and ground reaction forces are applied to the biped. The problem of design of ballistic swing motion is reduced to the boundary-value problem for the system of nonlinear differential equations with given initial and final configurations and the duration of the half-step. It is assumed that there is no friction in the interlink joints (note that the friction in the human joints is very small). Therefore, in the ballistic swing motion the complete energy of the system (kinetic energy plus potential one) is conserved and the system has the energy integral. Due to this fact some properties of symmetry of ballistic motions are proved. Linearized model can be reduced to the canonical Jordan form. Then the linear boundary-value problem can be solved analytically. Using numerical investigations of the linear model we have animated the biped walking, which occurs "similar" to the human gait: the transferring leg moves over the support, the legs bend with knees forward, and the trunk makes one vibration during one half-step. All these features have not been prescribed beforehand in the statement of the problem. Iteration process is used to solve the nonlinear boundary-value problem. For some cases, several solutions of this nonlinear problem are found numerically. The symmetry properties of the ballistic motions help to find numerically the solutions of this complex nonlinear problem. The ballistic motion is also designed numerically for the three-dimensional biped model with 6, 8, 9, and finally with 11 degrees of freedom. The double-support phase is assumed an instantaneous one. During this phase there is a collision of the transferring (swing) leg and the support. Active impulsive torques are applied in the interlink joints at this instant. These impulsive torques and ground reaction forces are described by delta functions of Dirac. Thus, with this impulsive control, most of the efforts are applied in the double-support phase. Formulas for the energy cost of impulsive control actions have been found. Some problems of optimal distribution of the impulsive actions between the joints are discussed. A planar seven-link model of biped with arms is considered. This model consists of a one-link trunk, two identical one-link arms, and two identical two-link legs with point feet. Ballistic gait of this biped model is studied. The goal of this study is to find the optimal amplitude of the arms swinging, minimizing the energy consumption. We show numerically the existence of the optimal amplitude of the arms swinging. Ballistic walking of a quadruped is investigated too. Three types of quadruped gaits, bound, amble, and trot, are considered. For each kind of the gait, ballistic locomotion is designed and energy consumption is evaluated.

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Introduction

According to some authors [see, for example, Mochon (1981); Mochon and McMahon (1980); McMahon (1984); Vitenzon (1998)], human and animal motions comprise alternating periods of muscle activity and relaxation. Perhaps for this kind of motion less energy is consumed. Furthermore, they suppose that during human walking, most of the efforts take place in the double-support phase. Besides, in the human gait the time of the double support is less than 20% of the whole step period.

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Taking into account the opinion mentioned above, we study locomotion with purely ballistic swing phases and instantaneous double-support phases. It means there are no active torques in the interlink joints during the single support motion. But at the instant of the double support, the impulsive torques are applied in the joints. These impulsive efforts are described by Dirac delta functions.

The problem of the ballistic swing motion design is reduced to the boundary-value problem for the equations, describing the single support phase. The analysis of the infinitesimal double support is reduced to the study of the algebraic equations, describing the jumps of the velocities under impulsive efforts. Our approach can be considered as an asymptotic one because the impulsive torques, describing by delta functions of Dirac, cannot be realized in practice.

The problem of the ballistic motion design for a biped has been stated by Formal'skii et al. (1975, 1977, 1978, 1980) and Formal'skii (1978). Later it has been considered in a number

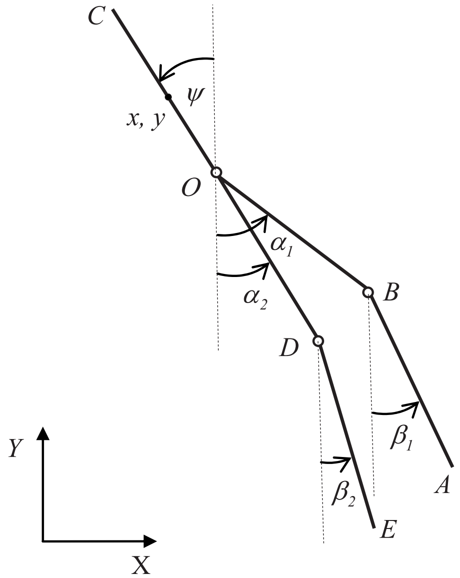

Fig. 1. Scheme of the biped

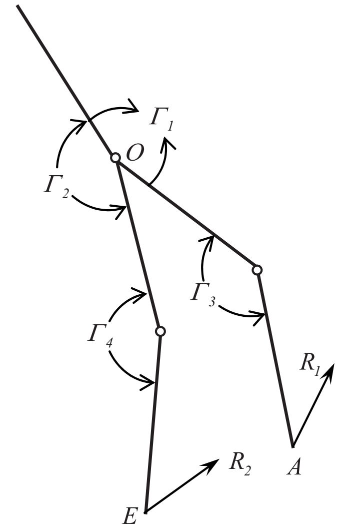

Fig. 2. Torques and forces

of papers [see, for example, Mochon (1981); Mochon and McMahon (1980); McMahon (1984); Formal'skii (1982, 1994a,b, 1997); McGeer (1989, 1990); Aoustin and Formal'skii (2008)]. In the monograph (Formal'skii 1982), the results of the papers (Formal'skii et al. 1975, 1977, 1978, 1980; Formal'skii 1978) have been summarized and extended. Using ballistic motions, it is shown in Aoustin and Formal'skii (2008) that the optimal amplitude of arms swinging exists and it is found numerically. The problem of ballistic walking design of a quadruped is studied in Formal'sky et al. (2000) for different gaits.

Equations-of-Motion of Five-Link Biped

We consider the biped walking in the sagittal plane. The planar five-link model contains the torso OC and two identical legs OBA and ODE (see Fig. 1). Each leg consists of the thigh (OB and OD) and the shin (BA and DE). All these five links are massive and absolutely rigid. Neglecting the friction in the interlink joints we assume them as ideal. This assumption comes from the fact that the friction in the human joints is very small.

Seven generalized coordinates x, y, ψ, α_1, α_2, β_1, and β_2 describe the position of the biped in the plane XY (Fig. 1). Here x and y are the Cartesian coordinates of the mass center of the body OC. Let Γ_1 and Γ_2 be the torques acting between the torso and the thighs, Γ_3 and Γ_4 be the torques applied in the knee joints, and $R_1(R_{1x},R_{1y})$ and $R_2(R_{2x},R_{2y})$ be the forces applied to the leg tips A and E (Fig. 2). The walk modeled here consists of the alternating phases of single and double supports. In the double-support phase, both legs are on the bearing surface (on the ground) and the forces R_1 and R_2 are the ground reaction forces. During the single support motion, one of the reaction forces R_1 or R_2 is the ground reaction, but the other one equals zero.

Using the second Lagrange method we come to the motion equations of the biped. Omitting the intermediate calculations, we can submit these equations in a visible matrix form

$$B(z)\ddot{z} + gF\|\sin z_i\| + D(z)\|\dot{z}_i^2\| = C(z)Q \qquad (1)$$

Here

$$z = \|z_i\| = \left\| \begin{matrix} x \\ y \\ \psi \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{matrix} \right\|, \quad \|\sin z_i\| = \left\| \begin{matrix} 0 \\ 1 \\ \sin\psi \\ \sin\alpha_1 \\ \sin\alpha_2 \\ \sin\beta_1 \\ \sin\beta_2 \end{matrix} \right\|, \quad \|\dot{z}_i^2\| = \left\| \begin{matrix} 0 \\ 0 \\ \dot{\psi}^2 \\ \dot{\alpha}_1^2 \\ \dot{\alpha}_2^2 \\ \dot{\beta}_1^2 \\ \dot{\beta}_2^2 \end{matrix} \right\|, \quad Q = \left\| \begin{matrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \end{matrix} \right\|$$

$B(z)$=symmetrical and positively definite inertia matrix of the size (7×7); F=diagonal (7×7) constant matrix of the potential energy; $D(z)$ and $C(z)$=matrices of the size (7×7) and (7×10), respectively; and g=gravity acceleration. The matrices $B(z)$, F, $D(z)$, and $C(z)$ are not presented here because of their inconvenience. These matrices can be found in Formal'skii et al. (1975, 1977, 1978, 1980) and Formal'skii (1978, 1982).

We have to add to system (1) the conditions for fixation of the supporting point foot in order to obtain a model of the swing motion. But to obtain this model we can also exclude the ground reaction force R_2 for the supporting leg from system (1) and put for the swing leg $R_1=0$. The scheme of the biped in the swing motion is shown in Fig. 3.

In the swing motion, the point foot E is motionless. We assume that it is connected with the bearing surface by an ideal joint. But during the locomotion of a human or walking robot the bearing surface does not keep the supporting legs. Thus, the ground reaction in the point foot E must be directed upward. In the swing motion, the biped has five degrees of freedom and five generalized coordinates: ψ, α_1, α_2, β_1, and β_2 (see Fig. 3). Putting the reaction force in the swing leg equals zero and excluding the ground reaction force of the supporting leg from system (1), we come to the equations of the swing motion in the following matrix form:

$$H(\zeta)\ddot{\zeta} + gL\|\sin\zeta_i\| + M(\zeta)\|\dot{\zeta}_i^2\| = N\Gamma \qquad (2)$$

Here

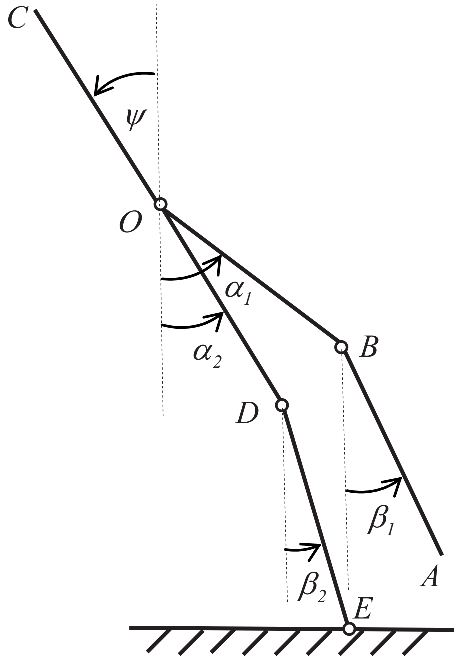

Fig. 3. Scheme of the biped in the single support

$$\zeta=\|\zeta_i\|=\left\|\begin{matrix}\psi\\ \alpha_1\\ \alpha_2\\ \beta_1\\ \beta_2\end{matrix}\right\|,\quad \|\sin\,\zeta_i\|=\left\|\begin{matrix}\sin\,\psi\\ \sin\,\alpha_1\\ \sin\,\alpha_2\\ \sin\,\beta_1\\ \sin\,\beta_2\end{matrix}\right\|,\quad \|\dot{\zeta}_i^2\|=\left\|\begin{matrix}\dot{\psi}^2\\ \dot{\alpha}_1^2\\ \dot{\alpha}_2^2\\ \dot{\beta}_1^2\\ \dot{\beta}_2^2\end{matrix}\right\|,\quad \Gamma=\left\|\begin{matrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\\ \Gamma_4\end{matrix}\right\|$$

The matrix $H(\zeta)$ of the kinetic energy is symmetrical and positively defined: $H(\zeta)=H^*(\zeta)>0$ (asterisk means transposition). The matrix L of the potential energy is a diagonal and constant one: $L=\mathrm{diag}\|l_{ii}\|=\mathrm{const}$. The matrix $M(\zeta)$ is an antisymmetric one: $M(\zeta)=-M^*(\zeta)$. All matrices $H(\zeta)$, L and $M(\zeta)$ are of the size (5×5). N is a constant matrix of the size (5×4). The matrices $H(\zeta)$ and $M(\zeta)$ depend on the differences between the angles ψ, α_1, α_2, β_1, and β_2, that is, on the interlink angles only. Matrix $H(\zeta)$ contains cosines of these differences; matrix $M(\zeta)$ contains sines. Having solved the matrix equation [Eq. (2)] we can check whether the vertical component of the ground reaction is directed upward and the supporting leg does not take off.

Problem Statement

The statement of the problem contains two parts. The one part concerns the swing motion; the other part is about the double-support phase.

Swing Motion

We assume that there are no active torques in the interlink joints during the swing motion. Therefore, we call this motion *ballistic*. However, active impulsive torques are applied in the instantaneous double-support phase. Thus, we consider ballistic (passive) swing motion with $\Gamma(t)\equiv 0$. The matrix equation of this ballistic single support motion can be obtained from Eq. (2)

$$H(\zeta)\ddot{\zeta}+gL\|\sin\,\zeta_i\|+M(\zeta)\|\dot{\zeta}_i^2\|=0 \tag{3}$$

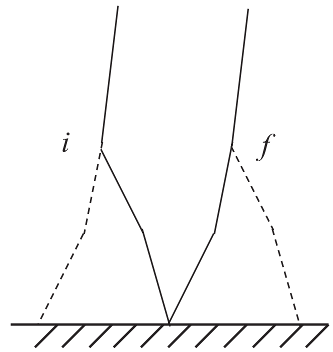

Fig. 4. Boundary configurations of the biped in the swing phase: *i*—initial configuration; *f*—final configuration

Let the biped at the start of the step $(t=0)$ be in the initial configuration shown in the left-hand side of Fig. 4 (position *i*). It is described by the vector

$$\zeta(0)=\|\psi(0),\alpha_1(0),\alpha_2(0),\beta_1(0),\beta_2(0)\|^* \tag{4}$$

In this initial configuration, the front and hind legs are both placed on the support.

At the time $t>0$ the front leg remains on the support, while the hind leg is being transferred. We want to transfer the biped into the final configuration (at the end of the half-step) shown in the right-hand side of Fig. 4 (position *f*) in fixed time $t=T$. This final configuration is described by the vector

$$\zeta(T)=\|\psi(T),\alpha_1(T),\alpha_2(T),\beta_1(T),\beta_2(T)\|^* \tag{5}$$

To obtain the cyclic gait on the horizontal surface this final configuration should coincided with the initial one (at the start of the half-step) with swapped legs. It means that the final configuration [Eq. (5)] has to satisfy the following equalities:

$$\psi(T)=\psi(0)$$

$$\alpha_1(T)=\alpha_2(0),\quad \alpha_2(T)=\alpha_1(0)$$

$$\beta_1(T)=\beta_2(0),\quad \beta_2(T)=\beta_1(0) \tag{6}$$

In the absence of the active torques the biped can be transferred from the given initial pose [Eq. (4)] to the given final pose [Eq. (5)] by choosing five initial angular velocities of the links

$$\dot{\psi}(0),\ \dot{\alpha}_1(0),\ \dot{\alpha}_2(0),\ \dot{\beta}_1(0),\ \dot{\beta}_2(0) \tag{7}$$

So, it is required to find the suitable vector $\dot{\zeta}(0)=\|\dot{\psi}(0),\dot{\alpha}_1(0),\dot{\alpha}_2(0),\dot{\beta}_1(0),\dot{\beta}_2(0)\|^*$ of the initial angular velocities. Hence the problem of the ballistic swing motion design is reduced to the mathematical boundary-value problem for the matrix differential equation [Eq. (3)]. We want to find the solution $\zeta(t)$ of Eq. (3) under boundary conditions [Eqs. (4) and (5)]. After solving the boundary-value problem the vector $\dot{\zeta}(0)$ of the initial angular velocities becomes known. The vectors $\dot{z}(0)$ and $\dot{z}(T)$ of the initial and terminal velocities become known too.

In the statement of the problem, we do not ensure that the transferred leg tip moves above the support, the legs bend "knee forward," the body does not fall down, and the reaction force is directed upward. But we check these conditions in the solution of the boundary-value problem after solving it.

The first part of the problem statement is formulated above. This part concerns the swing motion. But after the single support

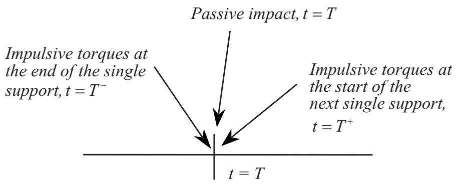

Fig. 5. Three subphases of the double-support phase

phase, the double support takes place. The second part of the problem statement concerns the instantaneous double support.

Instantaneous Double Support

In the human double-support phase, there is a transition of the support from one leg to another, and the duration of the double support is less than 20% of the whole step period. In our statement of the problem, the duration of the double support is assumed an infinitesimal one. Thus, the double support is assumed as instantaneous.

Let T be the instant when the double-support phase occurs. We divide this phase into three subphases. These subphases are presented in Fig. 5.

First Subphase

Let us assume that $T^-=T-0$ is the instant just before the front leg touches the ground. At the instant T^- the leg tip A does not touch the ground yet and the reaction force $R_1(T^-)=0$. Let us apply at the instant T^- impulsive torques in the four interlink joints, that is

$$\Gamma(t) = I^-\Delta(t - T^-) \tag{8}$$

Here $\Delta(t)$=Dirac delta function and

$$I^- = \|I_1^-, I_2^-, I_3^-, I_4^-\|^*$$

is the vector of the intensities of the impulsive actions (the weights of Dirac Δ functions). At the instant T^- the ground reaction force in the hind leg becomes also impulsive under the impulsive torques

$$R_{2x}(T^-) = I_{R_2x}^-\Delta(t - T^-), \quad R_{2y}(T^-) = I_{R_2y}^-\Delta(t - T^-)$$

Here $I_{R_2x}^-$ and $I_{R_2y}^-$=intensities of the two components of the ground reaction force. The vertical component $R_{2y}(T^-)$ of the ground reaction has to be directed upward. Consequently, the inequality $I_{R_2y}^- \geqslant 0$ has to take place, and it is necessary to check this inequality during the numerical investigations.

Let us integrate the matrix equation [Eq. (1)] on the infinitesimal time interval (T^-, T). Then we come to the matrix algebraic equation, which connects the vector $\dot{z}(T^-)$ of the velocities just before the impulsive action (8) and the vector $\dot{z}^a$ of the velocities just after the impulsive action (8)

$$B[z(T)][\dot{z}^a - \dot{z}(T^-)] = C[z(T)]I_Q^- \tag{9}$$

Here $z(T)$=given configuration of the biped at the end of the swing motion (in the double support); the vector $\dot{z}(T^-)$=known from the solution of the boundary-value problem; and

$$I_Q^- = \|I_1^-, I_2^-, I_3^-, I_4^-, 0, 0, I_{R_2x}^-, I_{R_2y}^-\|^*$$

The difference $\dot{z}^a - \dot{z}(T^-)$ is the jump of the velocity vector. The vector $\dot{z}^a$ contains *seven* unknown variables. For the swing phase with supporting hind leg, the linear velocities $\dot{x}^a$ and $\dot{y}^a$ can be expressed through the angular velocities of the links. Thus, system (9) contains seven scalar equations with 11 unknown variables, $\dot{\psi}^a$, $\dot{\alpha}_1^a$, $\dot{\alpha}_2^a$, $\dot{\beta}_1^a$, $\dot{\beta}_2^a$, I_1^-, I_2^-, I_3^-, I_4^-, $I_{R_2x}^-$, and $I_{R_2y}^-$.

Second Subphase

At the instant T, just after the impulsive action (8), let a *passive* impact of our biped succeed. It means there are no active impulsive torques in the interlink angles at the instant T. At this instant a constraint is imposed on the transferred leg, that is, its tip A becomes stationary with respect to the support surface. It results in action of an impulsive ground reaction force

$$R_{1x}(T) = I_{R_1x}\Delta(t - T), \quad R_{1y}(T) = I_{R_1y}\Delta(t - T)$$

We suppose that the hind leg takes off and the reaction force $R_2(T)=0$. The velocities of the biped links change instantaneously at the instant T. Let $\dot{z}^b$ be the vector of the velocities just after this passive impact. Using the matrix motion equation [Eq. (1)], we come to the matrix algebraic equation, which connects the vector $\dot{z}^a$ of the velocities just before the passive impact and the vector $\dot{z}^b$ of the velocities just after this impact

$$B[z(T)][\dot{z}^b - \dot{z}^a] = C[z(T)]I_Q \tag{10}$$

Here

$$I_Q = \|0, 0, 0, 0, I_{R_1x}, I_{R_1y}, 0, 0\|^*$$

The vector $\dot{z}^b$ contains seven unknown variables. But for the swing phase with a supporting front leg, the linear velocities $\dot{x}^b$ and $\dot{y}^b$ can be expressed through the angular velocities of the links. Thus, system (10) contains seven scalar equations with seven unknown variables, $\dot{\psi}^b$, $\dot{\alpha}_1^b$, $\dot{\alpha}_2^b$, $\dot{\beta}_1^b$, $\dot{\beta}_2^b$, I_{R_1x}, and I_{R_1y}. After the second subphase with the passive impact the vertical component of the velocity of the hind leg tip E has to be directed upward. The inequality $I_{R_1y} \geqslant 0$ has to take place, it is necessary to check these conditions during the numerical investigations.

Third Subphase

After the passive impact the next single support motion starts. But at the beginning of this next swing motion, at the instant $t=T^+=T+0$ (just after the passive impact) we apply another impulsive actions in the four interlink joints

$$\Gamma(t) = I^+\Delta(t - T^+) \tag{11}$$

Here

$$I^+ = \|I_1^+, I_2^+, I_3^+, I_4^+\|^*$$

is the vector of the intensities of the impulsive actions. At the instant $T^+=T+0$ the ground reaction force in the front leg becomes also impulsive

$$R_{1x}(T^+) = I_{R_1x}^+\Delta(t - T^+), \quad R_{1y}(T^+) = I_{R_1y}^+\Delta(t - T^+)$$

under the impulsive torques [Eq. (11)]. Using the matrix equation [Eq. (1)], we come to the matrix algebraic equation, which connects the vector $\dot{z}^b$ of the velocities just after the passive impact and the vector $\dot{z}(T^+)$ of the velocities just after applying impulsive torques [Eq. (11)]

$$B[z(T)][\dot{z}(T^+) - \dot{z}^b] = C[z(T)]I_Q^+ \tag{12}$$

Here

$$I_Q^+ = \|I_1^+, I_2^+, I_3^+, I_4^+, I_{R_1x}^+, I_{R_1y}^+, 0, 0\|^*$$

Since we design periodic biped walking, the vector $\dot{z}(T^+)=\dot{z}(0)$. At the same time, the vector $\dot{z}(0)$ is known from the solution of the boundary-value problem for the swing motion. System (12) contains seven scalar equations with six unknown variables, I_1^+, I_2^+, I_3^+, I_4^+, $I_{R_1x}^+$, and $I_{R_1y}^+$.

The configuration of the biped does not change during these three subphases because they are instantaneous. The velocities of the links change only (instantaneously). Vectors $\dot{z}^a$ and $\dot{z}^b$ are two intermediate vectors of the velocities. Systems (9), (10), and (12) contain 21 scalar equations and 24 unknown variables, $\dot{\psi}^a$, $\dot{\alpha}_1^a$, $\dot{\alpha}_2^a$, $\dot{\beta}_1^a$, $\dot{\beta}_2^a$, I_1^-, I_2^-, I_3^-, I_4^-, $I_{R_2x}^-$, $I_{R_2y}^-$, $\dot{\psi}^b$, $\dot{\alpha}_1^b$, $\dot{\alpha}_2^b$, $\dot{\beta}_1^b$, $\dot{\beta}_2^b$, I_{R_1x}, I_{R_1y}, I_1^+, I_2^+, I_3^+, I_4^+, $I_{R_1x}^+$, and $I_{R_1y}^+$. Therefore, this system has an infinite number of solutions. It is possible to choose a single solution by minimizing some function of the torque intensities. The vertical components of the reaction forces have to be directed upward. In addition, the vertical component of the velocity of the leg tip, which takes off, has to be directed upward.

Energy of the Impulsive Torques

Let us evaluate the energy expenses A_i of the control torque $\Gamma_i(t)$ $(i=1,2,3,4)$, acting on the time interval $[t_1,t_2]$, by the following integral:

$$A_i = \int_{t_1}^{t_2} |\Gamma_i(t)\dot{\delta}_i(t)|dt \tag{13}$$

Here δ_i $(i=1,2,3,4)$=interlink angle, which is equal to the difference between the corresponding absolute angles. Let torque $\Gamma_i(t)$ be an impulsive one

$$\Gamma_i(t) = I_i\Delta(t-\tau) \tag{14}$$

Here I_i=intensity of the impulsive action and $\Delta(t-\tau)$=Dirac delta function, which does not equal to zero at some instant, $t_1<\tau<t_2$, only. Then the energy expenses A_i of the impulsive torque [Eq. (14)] can be calculated by the following formula:

$$\begin{aligned} A_i &= \int_{\tau-0}^{\tau+0} |\Gamma_i(t)\dot{\delta}_i(t)|dt \\ &= \frac{I_i}{2}\begin{cases} |\dot{\delta}_i(\tau-0)+\dot{\delta}_i(\tau+0)| & \text{if } \dot{\delta}_i(\tau-0)\dot{\delta}_i(\tau+0) \geqslant 0 \\ \dfrac{\dot{\delta}_i^2(\tau-0)+\dot{\delta}_i^2(\tau+0)}{|\dot{\delta}_i(\tau+0)-\dot{\delta}_i(\tau-0)|} & \text{if } \dot{\delta}_i(\tau-0)\dot{\delta}_i(\tau+0) < 0 \end{cases} \end{aligned} \tag{15}$$

Using expression (15) we can calculate the energy expenses A corresponding to the control torques $\Gamma_i(t)$ $(i=1,2,3,4)$

$$A = \sum_{i=1}^{4} A_i \tag{16}$$

Minimizing function (16) with respect to the torque intensities, it is possible to find the optimal distribution of the impulsive torques in the biped joints at the infinitesimal double-support phase.

Properties of Symmetry of the Ballistic Swing Motion

System (3) of the equations of the ballistic swing motion is conservative (it has energy integral). Due to this fact its solutions have the following symmetry properties.

1. If the function $\zeta(t)$ is a solution of system (3), then the function $-\zeta(t)$ is a solution of this system as well. The configuration $-\zeta(t)$ is symmetric to the configuration $\zeta(t)$ with respect to the vertical axis, passing through the ankle joint of the supporting leg. It means that if the function $\zeta(t)$ describes the forward locomotion of the biped, then the function $-\zeta(t)$ describes the same locomotion of the biped but in opposite direction.
2. If the function $\zeta(t)$ is a solution of system (3), then the function $\zeta(-t)$ is a solution of this system as well. Each mechanical conservative system with even Lagrange function has this property. It is easy to prove properties in Items 1 and 2 by substituting the functions $-\zeta(t)$ and $\zeta(-t)$ into system (3). Along with the solution $\zeta(t)$ system (3) has the solution $\zeta(T+t)$ because this system is an autonomous one. Consequently, we obtain from Item 2 the following property.
3. If the function $\zeta(t)$ is a solution of system (3), then the function $\zeta(T-t)$ is a solution of this system as well. This property means particularly the following. If the function $\zeta(t)$ describes the locomotion of our biped model from the given configuration $\zeta(0)$ to the given final configuration $\zeta(T)$, then the function $\zeta(T-t)$ describes the locomotion from the configuration $\zeta(T)$ as the initial one back to the configuration $\zeta(0)$ as the final one (with the same stick diagram). From Items 1 and 3 the next property follows.
4. If the function $\zeta(t)$ is a solution of system (3), then the function $-\zeta(T-t)$ is a solution of this system as well.

   Now let the boundary configurations [Eqs. (4) and (5)] be symmetrical with respect to the vertical axis, passing through the ankle joint of the supporting leg, i.e.

$$\zeta(0) = -\zeta(T) \tag{17}$$

   If the solution of the boundary-value problem [Eqs. (3)–(5)] exists and is unique, then, according to property in Item 4, we obtain the following identity:

$$\zeta(t) \equiv -\zeta(T-t) \tag{18}$$

   The stick diagram of the motion (sequence of the configurations), satisfying identity (18), is symmetrical with respect to the vertical axis, passing through the ankle joint of the supporting leg. It follows from identity (18) that:

$$\zeta(T/2) = 0 \tag{19}$$

   Equality [Eq. (19)] means that at the instant $T/2$ all five links of the biped are placed on the vertical axis. If in the interval $0<t<T/2$ the transferred leg bends knee forward, then in the interval $T/2<t<T$ it bends "knee backward." Using property in Item 4, we can prove the following assertion.
5. The equality $\zeta(T/2)=0$ exists only for the symmetric solution $\zeta(t)$, satisfying identity (18). This symmetric solution can be designed by extending on the interval $[0,T]$ the solution with boundary conditions $\zeta(0)$ and $\zeta(T/2)=0$.

In Fig. 6, the stick diagram of the symmetrical solution of the nonlinear boundary-value problem [Eqs. (3)–(5)] is shown. This solution is obtained numerically for the five-link model with some anthropomorphic parameters. We have chosen these parameters for a person of mass $M=75$ kg and height of 1.75 m. In this solution, the half-step length $\Delta x=0.45$ m and its duration $T=0.5$ s. The boundary conditions are the following:

$$\psi(T) = \psi(0) = 0$$

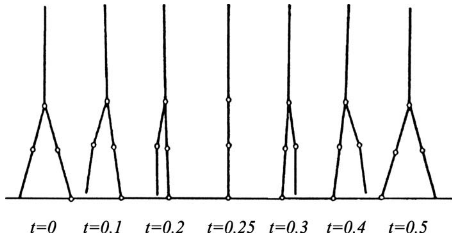

Fig. 6. Stick diagram of *symmetric* solution of the nonlinear boundary-value problem

$$\alpha_1(T)=\beta_1(T)=\alpha_2(0)=\beta_2(0),\quad \alpha_2(T)=\beta_2(T)=\alpha_1(0)=\beta_1(0) \tag{20}$$

These equalities mean that in the boundary configurations the torso is oriented strictly vertically and the legs are straight.

The gait shown in Fig. 6 does not look like human gait. All links of the biped at the middle time $t=0.25$ s of the half-step are oriented vertically. For the same boundary configurations [Eq. (20)], we have also found numerically another solution of the same nonlinear boundary-value problem—with the same half-step length $\Delta x=0.45$ m, the same boundary configurations, and time duration $T=0.5$ s. This solution is an asymmetric one. In Fig. 7, the stick diagram of this solution is shown.

The gait with this stick diagram looks like human gait; because the transferring leg moves over the support, it bends the knee forward, the supporting leg remains almost straight all the time, and the torso makes just one oscillation close to the vertical (with small amplitude).

According to property 4 the function $-\zeta(T-t)$ is also a solution of the same boundary-value problem (with the same boundary conditions). This solution describes the backward walking. So, with given above boundary configurations [Eq. (20)], the nonlinear boundary-value problem [Eqs. (3)–(5)] has at least three solutions. For the boundary configurations, which are close to the considered above [Eq. (20)], this problem has at least three solutions as well.

Linearized Boundary-Value Problem

Let us linearize system (3) near the equilibrium state $\zeta=0$

$$H(0)\ddot{\zeta}+gL\zeta=0 \tag{21}$$

The resulting linear autonomous system (21) is conservative. Therefore, using a linear nonsingular transformation with constant matrix R ($p=\|p_i\|$ is a new unknown vectorial variable with $i=1,\ldots,5$)

$$\zeta=Rp \tag{22}$$

we reduce this system to the canonical form (to normal coordinates) (Chetaev 1961, 1989)

$$\ddot{p}_i+\omega_i^2 p_i=0\quad (i=1,2) \tag{23}$$

$$\ddot{p}_i-\omega_i^2 p_i=0\quad (i=3,4,5) \tag{24}$$

Here

$$\lambda_i=\omega_i^2\quad (i=1,2)\text{ and }\lambda_i=-\omega_i^2\quad (i=3,4,5)$$

are the roots of the characteristic equation

$$\det\|H(0)\lambda-gL\|=0 \tag{25}$$

The characteristic equation [Eq. (25)] has two positive $\lambda_i=\omega_i^2$ ($i=1,2$) and three negative $\lambda_i=-\omega_i^2$ ($i=3,4,5$) roots because our model with five links has three inverted links—the thigh, the shin of the supporting leg, and the torso.

The boundary configurations for new systems (23) and (24) can be calculated with the following formulas:

$$p(0)=R^{-1}\zeta(0),\quad p(T)=R^{-1}\zeta(T) \tag{26}$$

The formulation of the boundary-value problem for systems (23) and (24) is the following. It is required to design the solution $p(t)$ of systems (23) and (24) with the boundary conditions [Eq. (26)]. Obviously, the required solution $p(t)$ has the form

$$p_i(t)=\frac{p_i(T)\sin\omega_i t+p_i(0)\sin[\omega_i(T-t)]}{\sin\omega_i T}\quad (i=1,2) \tag{27}$$

$$p_i(t)=\frac{p_i(T)\sinh\omega_i t+p_i(0)\sinh[\omega_i(T-t)]}{\sinh\omega_i T}\quad (i=3,4,5) \tag{28}$$

Solutions [Eqs. (27) and (28)] of Eqs. (23) and (24) with any boundary conditions [Eq. (26)] exist and are unique if and only if

$$\omega_i T\neq\pi k\quad (i=1,2;\ k=1,2,3,\ldots)$$

It is possible to go back to the original variables using transformation (22).

Thus, formulas (27) and (28) describe analytically the solution of the linear boundary-value problem. But it is necessary to carry out the numerical investigations in order to clarify the properties of the biped gait. Numerical solutions are obtained for the model with anthropomorphic parameters with different step lengths Δx, its duration times T, and boundary configurations. We have used the animation procedure to estimate the pattern of the synthesized locomotion. It is possible to choose the boundary configurations [Eqs. (4) and (5)] and times T so that the corresponding gaits in some sense look like human gait.

Nonlinear Boundary-Value Problem

In contrast to the linear boundary-value problem, the solution of the nonlinear boundary-value problem can be found using an iterative numerical procedure only. It is important to choose suitable initial approximation for the angular velocities $\dot{\psi}(0)$, $\dot{\alpha}_1(0)$, $\dot{\alpha}_2(0)$, $\dot{\beta}_1(0)$, and $\dot{\beta}_2(0)$ since, in other case, the iterations do not converge or converge very slow.

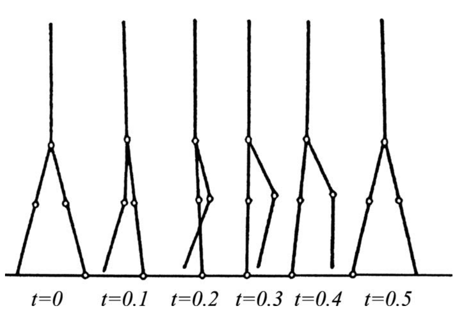

Fig. 7. Stick diagram of *asymmetric* solution of the nonlinear boundary-value problem

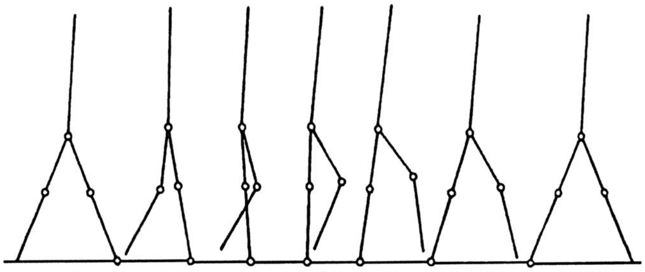

Fig. 8. Stick diagram with $\Delta x = 0.7$ m, $T = 0.55$ s, and $\psi(0) = \psi(T) = -0.075$

In Fig. 8, the stick diagram of the gait, corresponding to the solution of the nonlinear problem for the half-step length $\Delta x = 0.7$ m, duration $T = 0.55$ s, and torso inclination (initial and final) $\psi(0) = \psi(T) = -0.075$, is shown. (Recall that in this paper, all parameters are chosen for a person of mass $M = 75$ kg and height of 1.75 m.) The legs in the boundary configurations are straightened. One can see from Fig. 8 that the transferred leg moves above the support with the knee moving forward $[\alpha_2(t) > \beta_2(t)]$. The supporting leg remains all the time almost straight $[\alpha_1(t) \cong \beta_1(t)]$. The torso does not fall down making one oscillation near the vertical. For this case, the graphs of the horizontal $R_{1x}(t)$ and vertical $R_{1y}(t)$ components of the ground reaction force of the supporting leg are shown in Fig. 9. The vertical component R_{1y} of the ground reaction force is always positive during the walking. Its maximal value is less than the biped weight Mg. Do not forget that at the boundaries of the swing motion, impulsive ground reaction forces are applied to the biped. Due to this fact the theorem about the momentum is not violated. The sign of the horizontal component R_{1x} changes once per half-step as in human walking. Initially it is directed against the biped motion and then along the motion.

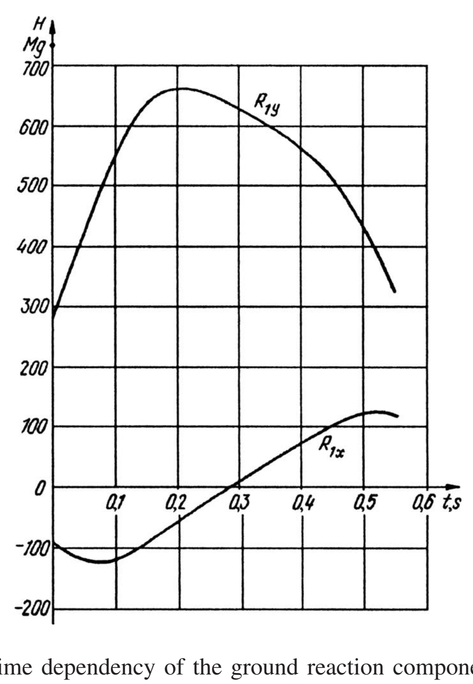

Fig. 9. Time dependency of the ground reaction components in the swing motion

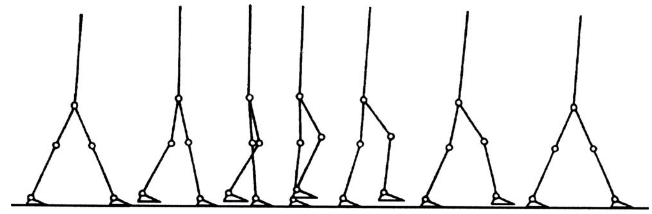

Fig. 10. Stick diagram of the walking of the seven-link model with massless feet

Planar Seven-Link Model with Massless Feet

The statement of the problem for the seven-link model is similar to the one for the five-link biped model without feet. Similarly the problem of the swing motion design is formulated mathematically as a boundary-value problem.

In the swing phase, the foot of the supporting leg is motionless; i.e., it is in equilibrium. But the ground reaction force is applied to the foot of the supporting leg, and in order to keep this foot in equilibrium we apply the torque in the ankle joint. This torque has to compensate the torque due to the reaction force. If the foot is massless, then its equations-of-motion become equilibrium conditions. The torques in the knee and hip joints are zero during the swing motion similarly to the five-link model. Thus, the motion of the seven-link model with the feet is not completely ballistic. We have investigated the nonlinear boundary-value problem for the seven-link model. Fig. 10 shows the stick diagram of the gait generated for the model with the feet. The boundary conditions are the same as above: $\Delta x = 0.7$ m, $T = 0.55$ s, and $\psi(0) = \psi(T) = -0.075$. The legs in the boundary configurations are straightened.

Optimal Amplitude of Arms Swinging for Biped Walking

In this section, we consider planar model of a biped without feet (with point feet) but with arms. This model consists of seven links: a one-link trunk, two identical two-link legs, and two identical one-link arms. The dynamic equations for this model are designed in Lagrange form. The structure of these equations is similar to the structure of Eq. (1) or (2) (for the single support). Ballistic gait of this biped model is studied (Aoustin and Formal'skii 2008). The structure of the dynamic equations in the ballistic swing motion is similar to the structure of Eq. (3). To find the ballistic single support motion we assign the half-step duration T and the boundary configurations of the biped (Fig. 11). The final configuration (at the end of the half-step) coincides with the initial one (at the start of the half-step), but the legs and arms are swapped. In these configurations, the arms are deflected from the vertical. The problem of ballistic single support motion design is reduced to the boundary-value problem. To obtain a cyclic gait the seven-link biped is controlled via corresponding impulsive torques at the instantaneous double support. These impulsive torques are applied in six interlink joints: knee, hip, and shoulder joints. For given boundary configurations and the half-step duration T, infinity of the solutions exists to find the intensities of the impulsive torques (as for the five-link biped). The unique solution is chosen by minimization the energy cost function. This function is calculated using formulas (15) and (16).

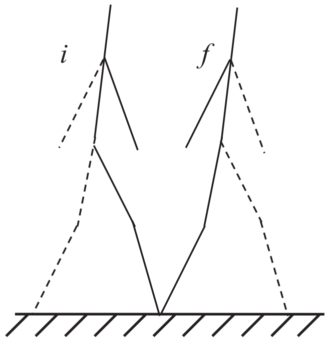

Fig. 11. Boundary configurations of the biped with one-link arms: i—initial configuration; f—final configuration

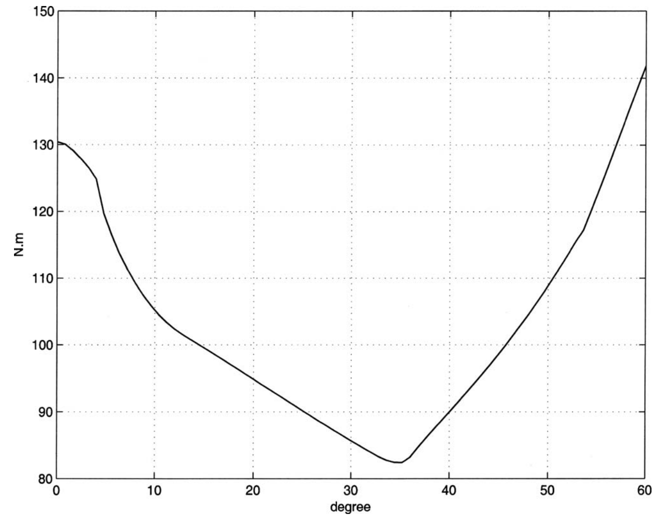

Fig. 12. Energy cost versus amplitude motion of the arms

The goal of this study is to find the optimal amplitude of the arms swinging, in other words to find the optimal deflection of the arms in the boundary configurations. An optimization criterion is the energy consumption. The physical parameters for a biped are chosen from average human data for a person of mass $M=75$ kg and height of 1.75 m. In Fig. 12, the energy consumption (in N·m) as a function of the half-amplitude (in degrees) of the arms swinging is shown. The results are obtained for the half-step length $\Delta x=0.45$ m and duration $T=0.45$ s. We see from Fig. 12 that with amplitude $2\times 35°$ the energy consumption is minimal. It means that this amplitude is optimal for the chosen half-step length and duration.

Thus, using ballistic trajectories and impulsive control, we have shown numerically the existence of the optimal amplitude of the arms swinging. The energy consumption is minimal if the arms swing with this amplitude. The numerical study shows that if the velocity of the biped decreases, the optimal amplitude of the arms swinging increases.

Three-Dimensional Ballistic Walking of the Biped Model with Many Degrees of Freedom

The investigation, described in this section, has been executed together with Yuriy Zavgorodniy and Andriy Telesh from Otto von Guericke University in Magdeburg (Germany). First of all, we add to our planar five-link model (without feet and arms) one degree of freedom in the ankle joint of the supporting leg. The new generalized coordinate is the angle between the shin and the vertical in the frontal plane. This new degree of freedom let the biped model deviate from the sagittal plane and incline in the frontal plane. Thus, due to this degree of freedom the motion becomes three-dimensional. Now the biped model in the swing motion has six degrees of freedom. To solve the new boundary-value problem we choose as initial approximation of the five angular velocities $\dot{\psi}(0)$, $\dot{\alpha}_1(0)$, $\dot{\alpha}_2(0)$, $\dot{\beta}_1(0)$, and $\dot{\beta}_2(0)$ the values from the solution of the boundary-value problem for the previous planar model with five degrees of freedom. In this case, the iterative process converges to the solution quickly.

Then we add one degree of freedom in each hip joint. With these two new degrees of freedom biped model in the swing motion has eight degrees of freedom. Due to these new degrees of freedom the transferring leg and the pelvis can be deflected in the frontal plane during the locomotion. To solve the new boundary-value problem we choose as initial approximation for the six angular velocities the values from the solution of the boundary-value problem for the previous three-dimensional (3D) model with six degrees of freedom.

Then we add to our model one degree of freedom between the trunk and the pelvis in the frontal plane. Now the biped model in the swing motion has nine degrees of freedom. Then we add two one-link arms. Each arm has one degree of freedom in the sagittal plane relative to the torso. Thus, we obtain finally a model with 11 degrees of freedom.

For each new model with the larger number of degrees of freedom we solve the corresponding boundary-value problem. As the initial approximation for the new boundary-value problem we use the initial angular velocities from the solution of the boundary-value problem for the previous model with less number of degrees of freedom. To see the locomotion of the model with 11 degrees of freedom we have animated this motion. The locomotion looks like human gait: the biped deviates little bit in the frontal plane, the transferring leg moves over the support, it bends the knee forward, the right and left arms move synchronously with the left and right legs, respectively, each arm oscillates relative to the torso once per half-step, and the torso slightly oscillates near the vertical.

It is difficult to design a mathematical model for the system with many degrees of freedom based on the Lagrange formalism. The Newton-Euler formalism (Schilling 1990) is used here for the models with many degrees of freedom.

Ballistic Walking Locomotion of a Quadruped

Ballistic locomotion of the walking quadruped on an even horizontal plane is investigated (Formal'sky et al. 2000). A quadruped model consists of a body (platform) and four identical two-link legs. Each leg of the quadruped model consists of a thigh and a shin. Each thigh is connected to the body by a one-degree-of-freedom rotating haunch joint and to the shin by a one-degree-of-freedom rotating knee joint [Fig. 13(a)]. The axes of all eight joints are parallel to the transverse axis of the quadruped's body. The body is symmetric with respect to its horizontal, longitudinal, and transverse planes passing through the geometric center of the body, which is also the center of mass of the body.

Three types of quadruped gaits, *bound*, *amble*, and *trot*, are studied. None of these gaits complies with a flight phase, but they

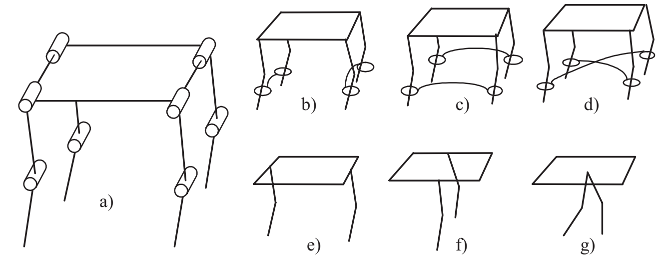

Fig. 13. Description of the studied quadruped and its gaits

all involve simultaneous and identical motion of two legs. The swing phase is ballistic, i.e., no active control torques are exerted. The ballistic motion is achieved due to appropriate choice of the initial velocities of the links. These velocities result from the impulsive active control torques and the ground reaction forces exerted at the boundary instants of the swing phase.

In the locomotion called *bound* here, both fore legs move identically with respect to the body and both hind legs move identically as well. This means that the fore legs (hind legs) of our quadruped are coupled [Fig. 13(b)]. We also suppose that at least two legs (fore or hind) are always on the support; our bound does not contain any flight phase. Obviously, a virtual leg can represent each pair of the coupled legs (Raibert et al. 1986). Thus, the model of the quadruped with coupled fore and coupled hind legs is equivalent to the model with two virtual legs [Fig. 13(e)].

For the *amble* gait, both right legs (both left legs) move always identically with respect to the body. This means that right legs (left legs) are coupled [Fig. 13(c)]. We also suppose that at least two legs (left or right) are always on the support. The simple model with two virtual legs considered for the amble gait is shown in Fig. 13(f).

For the *trot* gait, the diagonal legs move identically with respect to the body, as is shown in Fig. 13(d). Moreover, we suppose that either two or four legs are always on the support. A simple model with two virtual legs is introduced to study the trot gait [see Fig. 13(g)]. Each virtual leg is connected to the body at its center of mass by a one-degree-of-freedom rotating joint. The platform (body) of the quadruped for the trot gait with two virtual legs always remains horizontal.

Thus, for all three kinds of quadruped gaits the modeling of a four-legged robot gait is reduced to the modeling of a two-legged robot walking.

For each type of quadruped gait, the mathematical model is designed using the second Lagrange method. The boundary-value problem for the single support motion design is formulated. From a mathematical point of view, the design of the gaits is reduced to a boundary-value problem for the differential equations (describing the single support motion of the quadruped without active control torques) and the algebraic equations (describing the instantaneous phase between neighboring ballistic motions). Symmetry properties can be proved for the quadruped ballistic locomotion. These properties are similar to those formulated above for the biped ballistic locomotion. Iterative procedure is used to solve the nonlinear boundary-value problem. For all three kinds of quadruped gaits, the ballistic walking is designed. In Fig. 14, the stick diagram (sequence of the configurations) of two half steps of the quadruped for the bound gait is shown. These half steps are different.

The designed quadruped motions seem natural. All ballistic motions are admissible because the swing leg tip moves over the ground during the single support phase, and the reaction forces

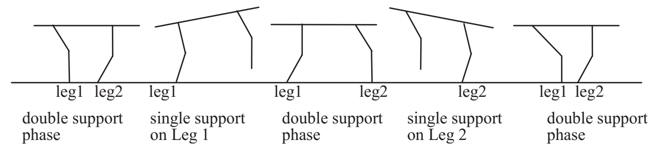

Fig. 14. Stick diagram of the quadruped's two half-steps for the bound gait

exerted by the ground satisfy the physical constraints. These properties are the intrinsic features of the quadruped ballistic motions; they are not inferred by the statement of the problem. The coast of the active impulsive torques like Eqs. (15) and (16) is lower for the amble and the trot than for the bound. It means that the amble and trot gaits are more efficient.

Conclusion

The problem of the ballistic locomotion design for a biped is formulated. In this problem statement, the locomotion is decomposed. First of all we design the single support motion considering the boundary-value problem. Then the impulsive control torques applied in the double support can be found.

The boundary-value problem is investigated both analytically and numerically. It has some symmetry properties, which are important to reveal the general pattern of the gait. For the linearized equations-of-motion the solution of the boundary-value problem is found analytically and it is almost always unique. For the complete nonlinear boundary-value problem a numerical solution can be found using an iterative procedure. Several solutions can be feasible for the nonlinear problem with some given boundary configurations.

In the numerical investigations, we have used anthropomorphic parameters. The boundary configurations and the duration of the step can be chosen such that the legs move over the support with the knee forward, the torso does not fall down and slightly oscillates near the vertical once per half-step. The vertical components of the ground reaction forces are directed upward both in single and in double-support phases. It is necessary to point that these "human" features of the gait are not prescribed by the problem statement. Hence they are intrinsic features of the ballistic motion. The designed ballistic gaits are in some sense humanlike. This resemblance gives additional hints to suppose that human walk contains some intervals of movement, which is close to ballistic.

The existence of the optimal amplitude of the arms swinging is shown numerically using biped ballistic locomotion.

Ballistic movements of the 3D biped model with many degrees of freedom are designed. Motion equations in the Newton-Euler form are used to solve the boundary-value problem for this model.

The problem of the ballistic locomotion design is stated and studied for the different paces of quadruped: bound, amble, and trot.

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