Kinematic Analysis and Computation of ZMP for a 12-internal-dof Biped Robot

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Abstract—*In this paper we present the kinematic analysis of a 12-internal-DOF (three-dimensional and anthropomorphic) biped robot, the Scout Lynxmotion©, towards a ZMP-based control. Since there are only few contributions that contain explicit models of similar robots, but not exactly this one, our first goal was to generate a kinematic-dynamic model that will let us study and control the locomotion of the biped. As the space is limited, this paper is restricted to the presentation of the representative steps of the kinematic modeling and the computation of the ZMP, which are then evaluated by numerical simulations. The dynamic model, which has been already obtained, will be published later on.*

Keywords: Biped robot, kinematics, mechanism, ZMP.

Nomenclature

θ_{ni}: Rotation angle of the shaft of the servomotor that trasmits movement to the ni link.
x_{ni}, y_{ni}, z_{ni}: Geometric parameters of the robot corresponding to the length of the ni link.
m_{ni}: Mass of the ni-th link.
$\mathbf{I^{mi}_{ni}}$: Inertia matrix of the ni-th link with respect to the frame mi.
$\mathbf{p_{ni}}$: Point ni.
$(\mathbf{O}; \mathbf{i_0}, \mathbf{j_0}, \mathbf{k_0})$: Reference inercial frame formed by mutually-orthogonal unit vectors $\mathbf{i_0}, \mathbf{j_0}, \mathbf{k_0}$, with origin at point $\mathbf{O}$.
$(\mathbf{p_{ni}}; \mathbf{i_{mi}}, \mathbf{j_{mi}}, \mathbf{k_{mi}})$: Reference local frame mi formed by mutually-orthogonal unit vectors $\mathbf{i_{mi}}, \mathbf{j_{mi}}, \mathbf{k_{mi}}$, with origin at point $\mathbf{p_{ni}}$.
$\mathbf{T_{zi}}$: Basic homogeneous transformation matrix.
$\mathbf{T_{mi,(m+1)i}}$: Homogeneous transformation matrix from reference frame mi to $(m+1)\,i$.
$\mathbf{r^{mi}_{ni}}$: Position vector ni with respect to reference frame mi.
$\mathbf{r^{mi}_{Gni}}$: Position vector of the center of gravity ni with respect to reference frame mi.
$\mathbf{R_x}$: Basic rotation matrix.
$\mathbf{R^{mi}_{(m+1)i}}$: Rotation matrix from base $(m+1)\,i$ to mi.
$\mathbf{v^{mi}_{ni}}$, $\omega^{mi}_{\mathbf{ni}}$, $\mathbf{a^{mi}_{ni}}$, $\alpha^{mi}_{\mathbf{ni}}$: Vectors of velocity, angular velocity, acceleration, angular aceleration ni with respect to reference frame mi, respectively.
$\mathbf{p_{ZMP}}$: Position vector associated to the ZMP.

Introduction

The study of biped walking machines was formally started during the 1970s and has rapidly increased in the recent years with different applications: developing of biped robots, exoskeletons, active ortheses, etc.

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In the literature of biped robots there are several contributions in different fields, for example in [1], [2], [3] general issues are reported such as the state of the art, definitions, guidelines to obtain models, discussions about mechanical design, description of walking patterns, etc.; in the field of control there are papers as [4], for trajectory planning some others like [5], [6], [7], simulation tools are given in [8] and so on.

However, in these contributions it is practically assumed that the reader already has the mathematical model, that is, they do not give it explicitly. For example, in the field of trajectory planning, [6] deals with the problem of controlling the ZMP for simplified inverted-pendulum models and then verifies the proposals on models of their own.

There are only few contributions that present explicit models, but none of them describe the movement of spatial 12 dof biped robots like the one presented here. For example, in [4] kinematic and dynamic (based on Newton-Euler formulation) models are given and [8] present dynamic equations for a two-dimensional biped robot. A major drawback of such contributions is that control schemes based on a Euler-Lagrange approach cannot be implemented.

Since our goal is to control the walk of a three-dimensional anthopomorphic biped robot, our first step is *to obtain both the kinematic and dynamic model using the Euler-Lagrange formulation of a biped robot Scout Lynxmotion© (Figure 1).*

In this paper only the kinematic model is presented.

Fig. 1. Robot de inters: Scout Lynxmotion ©

In order to verify numerically the obtained model, a first trajectory was tested. This trajectory was generated by instantaneous positions obtained from a CAD model. Nevertheless the main idea is to use the ZMP criterion of stability to generate the trajectories.

ZMP stands for Zero Moment Point, and can be briefly defined as a center of pressure of the floor reaction force [2]. As stability criterion, it is said that *if during the walking cycle the ZMP is contained in the support polygon, the walk is considered stable*. This criterion has established the theoretical basis of dynamic stability of locomotion and is applied in current versatile robots as those from MIT, Waseda University and Honda Company.

In Section I the simplified configuration of the robot is described. Based on this configuration the kinematic analysis of Section II is made, that is, the analysis of position, velocity and acceleration; because of space limitations, the last two were not completely developed. Based on some of these results, the computation of the ZMP is presented in Section III. Our model was verified by numerical simulations and the results are reported and discussed in section IV. Last section contains some conclusions and final remarks.

I. Biped robot architecture

Biped robot Scout consists of two 6-DOF serial link legs articulated between them by a central link, called torso because of its anatomical similarity to the human body. The 13 links are connected by rotational joints actuated by servomotors

In figure 2 the simplified CAD model of the robot is presented. That spatial configuration will be denoted as neutral position. Torso is identified with letter B and the links of each leg are labeled ni; where $1 \leq n \leq 6$ is determined by its position with respect to the torso, that is, $n = 1$ will denote the links immediatly jointed to it and $n = 6$ those who are playing the role of feet. The second index is used to distinguish each leg, that is, $i = 1$ are the links of the left leg and $i = 2$ those of the right leg.

The joints are presented as cylinders whose axes coincide with the shafts of the servomotors of the real model whereas the links are presented as bars. Rotations of the shafts of servomotors are denoted as θ_{ni}, where ni refers to the label of the link moved by the servo. According to the clasification proposed in [3], Scout is an anthropomorphical tridimensional biped robot, therefore a direct analogy with human anatomy is possible: the six red-colored angles are equivalent to the dofs of the hip, the two green-colored angles are meant to mimic the movement of the knee and the four blue-colored angles represent the dofs of the ankle. The geometry of the structure of the robot which has contact with the floor plane (which in fact is part of the links denoted as $6i$), has not been simplified since it will be important in the future when trying to walk with dynamic stability, in particular to force the ZMP to be placed within the support polygon.

Since the robot can move in 3-D, the coronal and sagittal views are required to define the geometric parameters with kinematic importance, which in this case are the distances between midpoints of the axes of the cylinders $\mathbf{p_{ni}}$ shown in Figure 3.

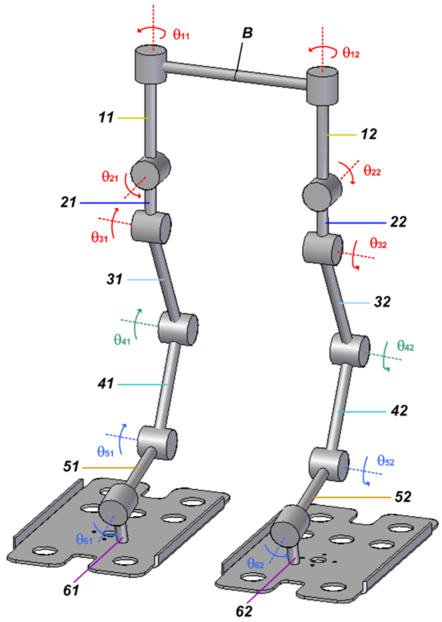

Fig. 2. Simplified model of the Scout.

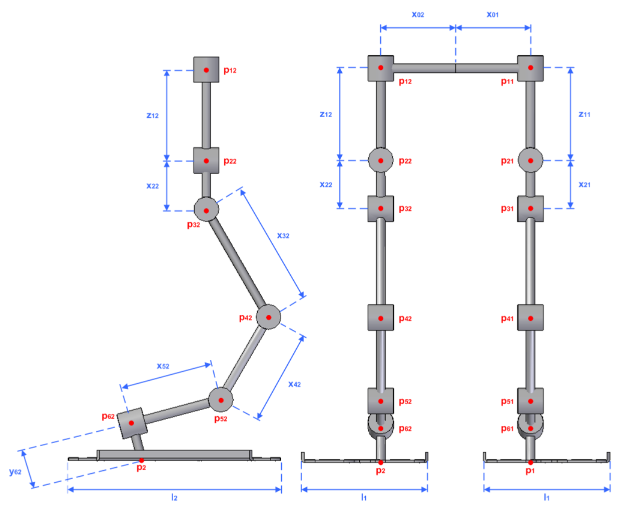

Fig. 3. Simplified model of the robot. Left: Sagittal right view. Right: Coronal anterior view.

It is important to note that points $\mathbf{p_i}$ are defined as the intersection of the lines perpendicular to the rotation axes θ_{6i} (which contain points $\mathbf{p_{6i}}$ in planes parallel to the sagittal plane) and the surfaces of the feet that make contact with the floor; distances between points $\mathbf{p_{6i}}$ and $\mathbf{p_i}$ along links $6i$ at the left ($i = 1$) and right ($i = 2$) feet are denoted as y_{6i}. Distances between the sagittal plane and parallel planes that contain the axes of the servomotors (fixed to the torso) are denoted as x_{0i}. In addition, l_1 and l_2 define the rectangled surface of contact between each feet and the floor.

Since the robot has sagittal symmetry the following equations can be defined: $x_{21} = x_{22}$, $x_{01} = x_{02}$, $x_{31} = x_{32}$, $z_{11} = z_{12}$, $x_{41} = x_{42}$, $y_{61} = y_{62}$, $x_{51} = x_{52}$.

These equations let us reduce the notation of geometric parameters to the following: $x_{0i}, z_{1i}, x_{2i}, x_{3i}, x_{4i}, x_{5i}, y_{6i}$.

II. Kinematic Equations

The kinematic model on which the walking analysis of the robot is based can be seen in Figures 4 and 5.

The origin $\mathbf{O}$ of the inertial fixed frame $(\mathbf{O}; \mathbf{i_0}, \mathbf{j_0}, \mathbf{k_0})$ and the unit vectors $\mathbf{i_0}, \mathbf{j_0}$ are located at the floor plane. Vector $\mathbf{i_0}$ is parallel to the anterior and posterior edges of the robot's feet, $\mathbf{j_0}$ is located at the intersection with the sagittal plane and $\mathbf{k_0}$ is the result of the cross product $\mathbf{i_0} \times \mathbf{j_0}$.

Point $\mathbf{p_B}$ of the local frame $(\mathbf{p_B}; \mathbf{i_B}, \mathbf{j_B}, \mathbf{k_B})$ is the middle point of the line defined from point $\mathbf{p_{11}}$ to point $\mathbf{p_{12}}$. Vectors $\mathbf{i_B}, \mathbf{j_B}$ are placed on the plane parallel to the superior face of link B that contains $\mathbf{p_B}$; $\mathbf{i_B}$ is parallel to the anterior and posterior edges of the same link and $\mathbf{j_B}$ is parallel to its lateral edges; finally, $\mathbf{k_B}$ is the cross product $\mathbf{i_B} \times \mathbf{j_B}$.

As mentioned before, $i = 1$ is defined for all elements of the left leg and $i = 2$ for those of the right leg, so it can be seen that points $\mathbf{p_i}$ of the local frames $(\mathbf{p_i}; \mathbf{i_i}, \mathbf{j_i}, \mathbf{k_i})$ are located at the plane of the feet-floor contact, as well as vectors $\mathbf{i_i}, \mathbf{j_i}$. Vectors $\mathbf{i_i}$ are parallel to the anterior and posterior edges of the corresponding foot, $\mathbf{j_i}$ are parallel to the lateral edges and $\mathbf{k_i}$ are the cross products $\mathbf{i_i} \times \mathbf{j_i}$.

Based on the defined reference frames, the inverse kinematic problem for the analysis of the position of the biped robot can de stated as:

Given a certain position of points $\mathbf{p_B}, \mathbf{p_i}$ *and the orientation of the local frames* $(\mathbf{p_B}; \mathbf{i_B}, \mathbf{j_B}, \mathbf{k_B}), (\mathbf{p_i}; \mathbf{i_i}, \mathbf{j_i}, \mathbf{k_i})$ *with respect to the inertial frame* $(\mathbf{O}; \mathbf{i_0}, \mathbf{j_0}, \mathbf{k_0})$*, find the angles* $\theta_{1i}, \theta_{2i}, \theta_{3i}, \theta_{4i}, \theta_{5i}, \theta_{6i}$ *such that the spatial configuration determined by those positions and orientations can be archieved.*

The position of points $\mathbf{p_B}, \mathbf{p_i}$ with respect to the intertial frame is described by the following vectors

$$\begin{aligned} \mathbf{r_B^0} &= x_B \mathbf{i_0} + y_B \mathbf{j_0} + z_B \mathbf{k_0} \\ \mathbf{r_i^0} &= x_i \mathbf{i_0} + y_i \mathbf{j_0} + z_i \mathbf{k_0} \end{aligned} \tag{1}$$

The orientation of the local frame $(\mathbf{p_B}; \mathbf{i_B}, \mathbf{j_B}, \mathbf{k_B})$ with respect to the inertial frame is defined by the following Euler angles:

$$\theta_B, \phi_B, \psi_B \tag{2}$$

For local frames placed at the feet $(\mathbf{p_i}; \mathbf{i_i}, \mathbf{j_i}, \mathbf{k_i})$ the corresponding Euler angles are:

$$\theta_i, \phi_i, \psi_i \tag{3}$$

Define the Euler angles θ_i, ϕ_i, ψ_i to describe the orientation of the local frames $(\mathbf{p_i}; \mathbf{i_i}, \mathbf{j_i}, \mathbf{k_i})$ with respect to the inertial frame. Angle θ_i corresponds to rotations about the $\mathbf{i_0}$ axis, ϕ_i represents rotation about the $\mathbf{j_{\theta i}}$ axis and ψ_i corresponds to rotation about the $\mathbf{k_{\phi i}}$ axis. Similarly, for the orientation of the local frame $(\mathbf{p_B}; \mathbf{i_B}, \mathbf{j_B}, \mathbf{k_B})$, θ_B corresponds to rotation about the $\mathbf{i_0}$ axis, ϕ_B describe rotation about the $\mathbf{j_{\theta B}}$ axis and ψ_B represents rotation about the $\mathbf{k_{\phi B}}$ axis.

By obtaining the time derivatives of position vectors (1) and Euler angles (2) - (3), the inverse kinematic problem of velocity is defined as follows:

Given certain translation and rotation velocities of the torso and feet links, that is, with vectors $\mathbf{v_B^0} = \dot{x}_B \mathbf{i_0} + \dot{y}_B \mathbf{j_0} + \dot{z}_B \mathbf{k_0}, \mathbf{v_i^0} = \dot{x}_i \mathbf{i_0} + \dot{y}_i \mathbf{j_0} + \dot{z}_i \mathbf{k_0}$ *and time derivatives of the Euler angles* $\dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{\theta}_i, \dot{\phi}_i, \dot{\psi}_i$*, find the joint velocities of the links of the robot* $\dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$.

By obtaining the time derivatives of velocity vectors $\mathbf{v_B^0}, \mathbf{v_i^0}$ and $\dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{\theta}_i, \dot{\phi}_i, \dot{\psi}_i$, the inverse kinematic problem of acceleration is then defined as:

Given certain translation and rotation accelerations of the torso and feet links, that is, with vectors $\mathbf{a_B^0} = \ddot{x}_B \mathbf{i_0} + \ddot{y}_B \mathbf{j_0} + \ddot{z}_B \mathbf{k_0}, \mathbf{a_i^0} = \ddot{x}_i \mathbf{i_0} + \ddot{y}_i \mathbf{j_0} + \ddot{z}_i \mathbf{k_0}$ *and second time derivatives of the Euler angles* $\ddot{\theta}_B, \ddot{\phi}_B, \ddot{\ }_B, \ddot{\theta}_i, \ddot{\phi}_i, \ddot{\ }_i$*, find the joint accelerations of the links of the robot* $\ddot{\theta}_{1i}, \ddot{\theta}_{2i}, \ddot{\theta}_{3i}, \ddot{\theta}_{4i}, \ddot{\theta}_{5i}, \ddot{\theta}_{6i}$.

In the next sections the answer to the three inverse kinematic problems are described, that is, some equations are presented in order to give numerical solution to those problems.

Regarding the number of input data required to solve the position problem, it is clear that the spatial configuration of the robot can only be defined by eighteen input values: nine Cartesian coordinates from (1), i.e. x_B, y_B, z_B, x_i, y_i, z_i, $i = 1, 2$, and nine Euler angles from (2) and (3), i.e. θ_B, ϕ_B, ψ_B, θ_i, ϕ_i, ψ_i, $i = 1, 2$. Therefore, it can be said that the robot has 18 dof.

On the other hand, in the specialized literature biped robots are usually characterized by its joint number, which in fact is identified as the number of internal degrees of freedom. From this point of view, the Scout robot is described as a robot with 12 internal dof.

According to literature, 12 is the minimum number of internal dof required in a biped robot to synthesize three-dimensional walking cycles similar to human locomotion [1].

A. Position equations

In order to solve the position inverse kinematic problem, the concept of basic homogeneous matrix transformations was used.

Transformations $\mathbf{T_{z1}}, \mathbf{T_{z2}}, \mathbf{T_{z3}}$ represent translations with displacements x, y, z defined with respect to axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively; transformations $\mathbf{T_{z4}}, \mathbf{T_{z5}}, \mathbf{T_{z6}}$ describe rotations with angular displacements $\theta_x, \theta_y, \theta_z$ about the same axes.

According to the Cartesian coordinates of points $\mathbf{p_B}, \mathbf{p_i}$ with respect to the intertial frame and the Euler angles in (2) and (3), the required transformations to express the position and the orientation of local frames of the torso $(\mathbf{p_B}; \mathbf{i_B}, \mathbf{j_B}, \mathbf{k_B})$ and the feet $(\mathbf{p_i}; \mathbf{i_i}, \mathbf{j_i}, \mathbf{k_i})$ with respect to the inertial frame as a combination of translations and rotations are the following:

$$\mathbf{T_{0,B}} = \mathbf{T_{z1}}(x_B)\mathbf{T_{z2}}(y_B)\mathbf{T_{z3}}(z_B)\mathbf{T_{z4}}(\theta_B)\mathbf{T_{z5}}(\phi_B)\mathbf{T_{z6}}(\psi_B)$$

$$\mathbf{T_{0,i}} = \mathbf{T_{z1}}(x_i)\mathbf{T_{z2}}(y_i)\mathbf{T_{z3}}(z_i)\mathbf{T_{z4}}(\theta_i)\mathbf{T_{z5}}(\phi_i)\mathbf{T_{z6}}(\psi_i)$$

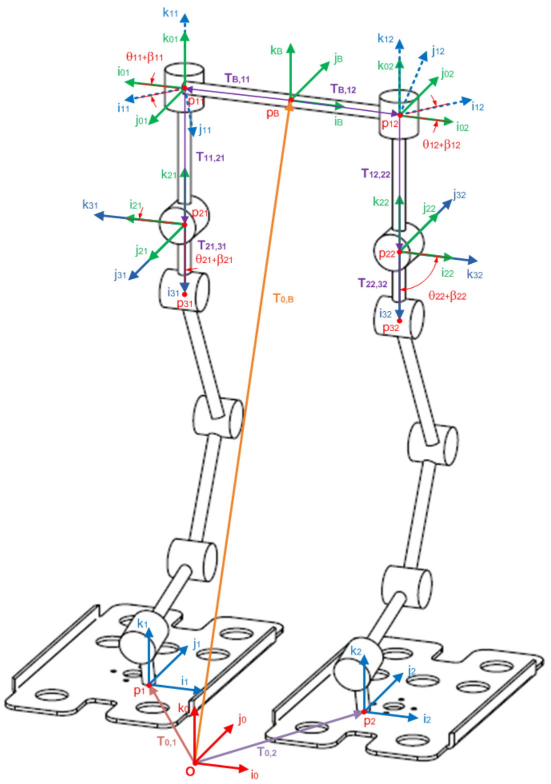

Fig. 4. Local frames used to describe the position equation by homogeneous transformations. The transformation path from one local frame into the next one is illustrated with violet-colored arrows.

Transformations $\mathbf{T_{ni,(n+1)i}}$, which are listed in the next paragraph, establish the product of the basic homogeneous transformations that translate and rotate the local frames placed at $\mathbf{p_{ni}}$ into those which are placed at $\mathbf{p_{(n+1)i}}$.

$$\begin{aligned}
\mathbf{T_{B,11}} &= \mathbf{T_{z1}}(-x_{01})\mathbf{T_{z6}}(\theta_{11}+\beta_{11}) \\
\mathbf{T_{B,12}} &= \mathbf{T_{z1}}(x_{02})\mathbf{T_{z6}}(\theta_{12}+\beta_{12}) \\
\mathbf{T_{1i,2i}} &= \mathbf{T_{z3}}(-z_{1i})\mathbf{T_{z5}}(\theta_{2i}+\beta_{2i}) \\
\mathbf{T_{2i,3i}} &= \mathbf{T_{z1}}(x_{2i})\mathbf{T_{z6}}(\theta_{3i}+\beta_{3i}) \\
\mathbf{T_{3i,4i}} &= \mathbf{T_{z1}}(x_{3i})\mathbf{T_{z6}}(\theta_{4i}+\beta_{4i}) \\
\mathbf{T_{4i,5i}} &= \mathbf{T_{z1}}(x_{4i})\mathbf{T_{z6}}(\theta_{5i}+\beta_{5i}) \\
\mathbf{T_{5i,6i}} &= \mathbf{T_{z1}}(x_{5i})\mathbf{T_{z6}}(\theta_{6i}+\beta_{6i}) \\
\mathbf{T_{6i,i}} &= \mathbf{T_{z2}}(y_{6i})\mathbf{T_{z6}}(\beta_7)\mathbf{T_{z5}}(\beta_8)
\end{aligned}$$

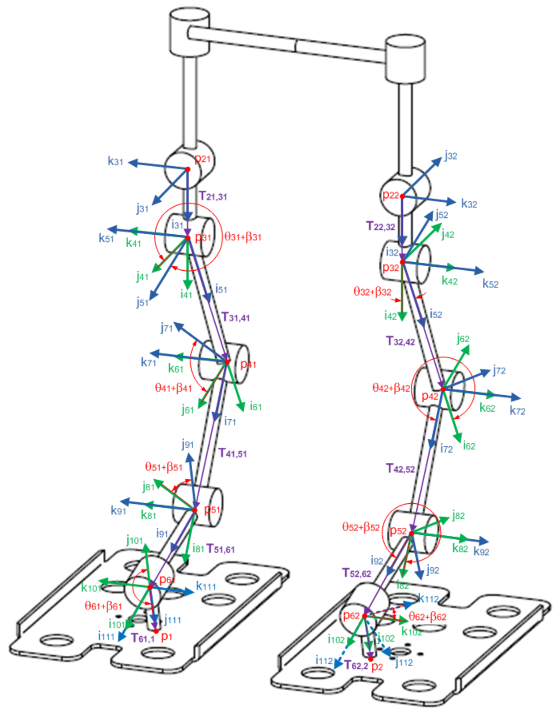

Fig. 5. Local frames used to describe the position equation by homogeneous transformations. Green-colored frames are obtained after a translation; blue-colored frames are generated after a rotation.

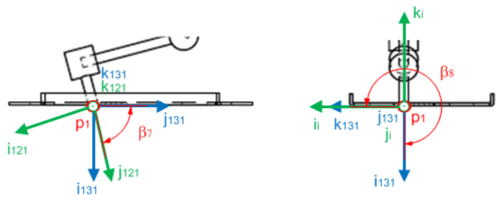

Fig. 6. Local frames placed at the feet, angles β_7 and β_8 are the same in both of them.

Transformations $\mathbf{T_{B,11}}, \mathbf{T_{B,12}}$ correspond to the alignment of the frame of the torso with local frames $(\mathbf{p_{1i}}; \mathbf{i_{1i}}, \mathbf{j_{1i}}, \mathbf{k_{1i}})$; transformation $\mathbf{T_{6i,i}}$ corresponds to the alignment of local frames $(\mathbf{p_{6i}}; \mathbf{i_{11i}}, \mathbf{j_{11i}}, \mathbf{k_{11i}})$ and $(\mathbf{p_i}; \mathbf{i_i}, \mathbf{j_i}, \mathbf{k_i})$, as it is illustrated in Figure 6. Angles β_{ni} are constant values defined in order to recover the so-called *neutral position*, that is, when the servomotors are also in neutral position: $\theta_{1i} = \theta_{2i} = \theta_{3i} = \theta_{4i} = \theta_{5i} = \theta_{6i} = 0$.

Taking into account that the product of the homogeneous transformations that define the translations and rotations between the inertial frames and the local frames all the way through the torso and the rotational joints of each leg must be equal to the product of the transformations in $\mathbf{T_{0,i}}$, the following matrix loop equation can be defined:

$$\mathbf{T_{0,B}}\mathbf{T_{B,1i}}\mathbf{T_{1i,2i}}\mathbf{T_{2i,3i}}\mathbf{T_{3i,4i}}\mathbf{T_{4i,5i}}\mathbf{T_{5i,6i}}\mathbf{T_{6i,i}} = \mathbf{T_{0,i}} \tag{4}$$

From matrix equation (4) one can pose a set of 12 scalar equations to solve the inverse kinematic problem of position for each one of the legs of the robot. By taking 3 equations from the 4^{th} column and 3 others from the diagonal of the spherical image a square system of equations can be defined, that is, 6 equations and 6 unknown variables which are the link-angles: $\theta_{11}, \theta_{21}, \theta_{31}, \theta_{41}, \theta_{51}, \theta_{61}$ for the left leg and $\theta_{12}, \theta_{22}, \theta_{32}, \theta_{42}, \theta_{52}, \theta_{62}$ for the right one.

In order to consider the footprints in walking simulations, the vectors that define the vertices of the corresponding polygons should be given. Those vectors are illustrated in Figure 7.

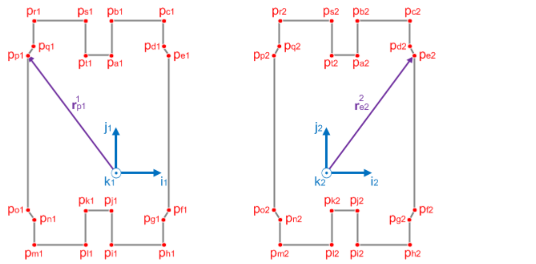

Fig. 7. Vertices of polygons that characterize the footprints. Vectors $\mathbf{r^1_{p1}}, \mathbf{r^2_{e2}}$ represent the position of points $\mathbf{p_{p1}}$ y $\mathbf{p_{e2}}$ in the local frames $(\mathbf{p_1}; \mathbf{i_1}, \mathbf{j_1}, \mathbf{k_1})$ and $(\mathbf{p_2}; \mathbf{i_2}, \mathbf{j_2}, \mathbf{k_2})$.

With vectors $\mathbf{r^i_{ai}}, \mathbf{r^i_{bi}}, \ldots, \mathbf{r^i_{ti}}$ (defined in appendix A), Cartesian coordinates of vertices with respect to the inertial frame can be computed $(ai_{x0}, ai_{y0}, ai_{z0})$, $(bi_{x0}, bi_{y0}, bi_{z0})$, $\ldots, (ti_{x0}, ti_{y0}, ti_{z0})$. If we consider as an example vertices $\mathbf{p_{ai}}$, one must compute:

$$\mathbf{T_{0,i}} \mathbf{T_{z1}}(ai_x) \mathbf{T_{z2}}(ai_y) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} ai_{x0} \\ ai_{y0} \\ ai_{z0} \\ 1 \end{bmatrix}$$

where ai_x, ai_y are the coordinates of vertices $\mathbf{p_{ai}}$ with respect to the axes i_i and j_i of the local frames, respectively.

B. Velocity equations

In order to get an efficient solution of velocity equations a vectorial method based on the position equations [9] was chosen

$$\mathbf{r^0_B} + \mathbf{r^0_{0i}} + \mathbf{r^0_{1i}} + \mathbf{r^0_{2i}} + \mathbf{r^0_{3i}} + \mathbf{r^0_{4i}} + \mathbf{r^0_{5i}} + \mathbf{r^0_{6i}} = \mathbf{r^0_i} \quad (5)$$

where the position vectors of Figure 8 with respect to the inertial frame are:

$$\begin{aligned} \mathbf{r^0_B} &= x_B \mathbf{i_0} + y_B \mathbf{j_0} + z_B \mathbf{k_0} \\ \mathbf{r^0_i} &= x_i \mathbf{i_0} + y_i \mathbf{j_0} + z_i \mathbf{k_0} \end{aligned}$$

$$\begin{aligned} \mathbf{r^0_{01}} &= -x_{01} \mathbf{i^0_B} & \mathbf{r^0_{02}} &= x_{02} \mathbf{i^0_B} \\ \mathbf{r^0_{1i}} &= -z_{1i} \mathbf{k^0_{1i}} & \mathbf{r^0_{2i}} &= x_{2i} \mathbf{i^0_{3i}} \\ \mathbf{r^0_{3i}} &= x_{3i} \mathbf{i^0_{5i}} & \mathbf{r^0_{4i}} &= x_{4i} \mathbf{i^0_{7i}} \\ \mathbf{r^0_{5i}} &= x_{5i} \mathbf{i^0_{9i}} & \mathbf{r^0_{6i}} &= y_{6i} \mathbf{j^0_{11i}} \end{aligned}$$

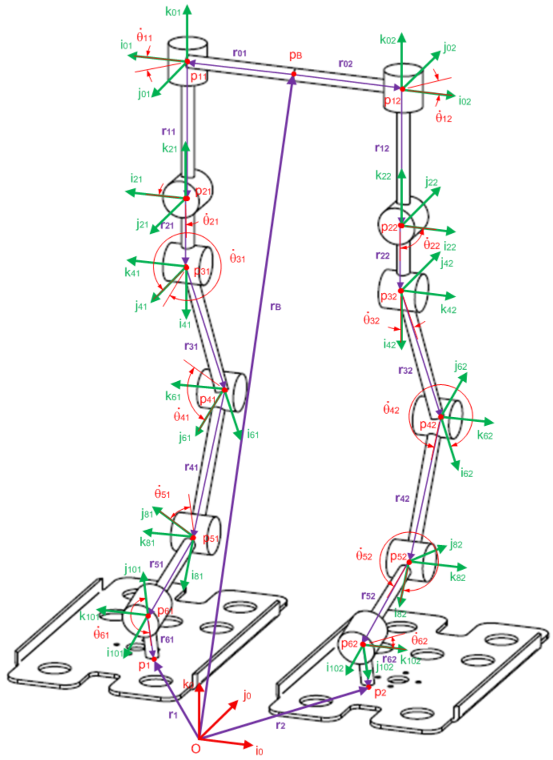

Fig. 8. Vectors used to define the velocity and acceleration equations (violet-colored).

Unit vectors $\mathbf{i_B}, \mathbf{k_{1i}}, \mathbf{i_{3i}}, \mathbf{i_{5i}}, \mathbf{i_{7i}}, \mathbf{i_{9i}}, \mathbf{j_{11i}}$ can be described with respect to the inertial frame by basic rotation matrices with angles $\theta_x, \theta_y, \theta_z$ about axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\begin{aligned} \mathbf{i^0_B} &= \mathbf{R_x}(\theta_B) \mathbf{R_y}(\phi_B) \mathbf{R_z}(\psi_B) \mathbf{i_B} \\ &= \mathbf{R^0_B} \mathbf{i_B} \\ \mathbf{k^0_{1i}} &= \mathbf{R^0_B} \mathbf{R_z}(\theta_{1i} + \beta_{1i}) \mathbf{k_{1i}} \\ &= \mathbf{R^0_{1i}} \mathbf{k_{1i}} \\ \mathbf{i^0_{3i}} &= \mathbf{R^0_{1i}} \mathbf{R_y}(\theta_{2i} + \beta_{2i}) \mathbf{i_{3i}} \\ &= \mathbf{R^0_{3i}} \mathbf{i_{3i}} \\ \mathbf{i^0_{5i}} &= \mathbf{R^0_{3i}} \mathbf{R_z}(\theta_{3i} + \beta_{3i}) \mathbf{i_{5i}} \\ &= \mathbf{R^0_{5i}} \mathbf{i_{5i}} \\ \mathbf{i^0_{7i}} &= \mathbf{R^0_{5i}} \mathbf{R_z}(\theta_{4i} + \beta_{4i}) \mathbf{i_{7i}} \\ &= \mathbf{R^0_{7i}} \mathbf{i_{7i}} \\ \mathbf{i^0_{9i}} &= \mathbf{R^0_{7i}} \mathbf{R_z}(\theta_{5i} + \beta_{5i}) \mathbf{i_{9i}} \\ &= \mathbf{R^0_{9i}} \mathbf{i_{9i}} \\ \mathbf{j^0_{11i}} &= \mathbf{R^0_{9i}} \mathbf{R_x}(\theta_{6i} + \beta_{6i}) \mathbf{j_{11i}} \\ &= \mathbf{R^0_{11i}} \mathbf{j_{11i}} \end{aligned}$$

Time derivative of position vector equation (5) is:

$$\dot{\mathbf{r}}^0_B + \dot{\mathbf{r}}^0_{0i} + \dot{\mathbf{r}}^0_{1i} + \dot{\mathbf{r}}^0_{2i} + \dot{\mathbf{r}}^0_{3i} + \dot{\mathbf{r}}^0_{4i} + \dot{\mathbf{r}}^0_{5i} + \dot{\mathbf{r}}^0_{6i} = \dot{\mathbf{r}}^0_i$$

If one define $\mathbf{v} = \dot{\mathbf{r}}$, the velocity vector equation for the translation of each leg is:

$$\mathbf{v_B^0} + \mathbf{v_{0i}^0} + \mathbf{v_{1i}^0} + \mathbf{v_{2i}^0} + \mathbf{v_{3i}^0} + \mathbf{v_{4i}^0} + \mathbf{v_{5i}^0} + \mathbf{v_{6i}^0} = \mathbf{v_i^0} \quad (6)$$

where the velocity vectors with respect to the inertial frame are:

$$\begin{aligned} \mathbf{v_B^0} &= \dot{x}_B\mathbf{i_0} + \dot{y}_B\mathbf{j_0} + \dot{z}_B\mathbf{k_0} \\ \mathbf{v_i^0} &= \dot{x}_i\mathbf{i_0} + \dot{y}_i\mathbf{j_0} + \dot{z}_i\mathbf{k_0} \end{aligned}$$

$$\begin{array}{lllllll} \mathbf{v_{0i}^0} & = & \boldsymbol{\omega}_B^0 \times \mathbf{r_{0i}^0} & & \mathbf{v_{1i}^0} & = & \boldsymbol{\omega}_{1i}^0 \times \mathbf{r_{1i}^0} \\ \mathbf{v_{2i}^0} & = & \boldsymbol{\omega}_{2i}^0 \times \mathbf{r_{2i}^0} & & \mathbf{v_{3i}^0} & = & \boldsymbol{\omega}_{3i}^0 \times \mathbf{r_{3i}^0} \\ \mathbf{v_{4i}^0} & = & \boldsymbol{\omega}_{4i}^0 \times \mathbf{r_{4i}^0} & & \mathbf{v_{5i}^0} & = & \boldsymbol{\omega}_{5i}^0 \times \mathbf{r_{5i}^0} \\ \mathbf{v_{6i}^0} & = & \boldsymbol{\omega}_{6i}^0 \times \mathbf{r_{6i}^0} & & & & \end{array} \quad (7)$$

In these equations angular velocity vector of the torso is denoted as $\boldsymbol{\omega}_B^0$ and $\boldsymbol{\omega}_{ni}^0$ represents the angular velocity vector of the *ni*-th link.

If vectors (7) are substituted in (6), one obtains:

$$\begin{aligned} &\dot{x}_B\mathbf{i_0} + \dot{y}_B\mathbf{j_0} + \dot{z}_B\mathbf{k_0} + \boldsymbol{\omega}_B^0 \times \mathbf{r_{0i}^0} + \boldsymbol{\omega}_{\mathbf{1i}}^{\mathbf{0}} \times \mathbf{r_{1i}^0} + \\ &+\boldsymbol{\omega}_{2i}^0 \times \mathbf{r_{2i}^0} + \boldsymbol{\omega}_{\mathbf{3i}}^{\mathbf{0}} \times \mathbf{r_{3i}^0} + \boldsymbol{\omega}_{\mathbf{4i}}^{\mathbf{0}} \times \mathbf{r_{4i}^0} + \\ &+\boldsymbol{\omega}_{5i}^0 \times \mathbf{r_{5i}^0} + \boldsymbol{\omega}_{6i}^0 \times \mathbf{r_{6i}^0} = \dot{x}_i\mathbf{i_0} + \dot{y}_i\mathbf{j_0} + \dot{z}_i\mathbf{k_0} \end{aligned} \quad (8)$$

Angular velocity equations for each leg are:

$$\begin{aligned} \boldsymbol{\omega}_{6i}^0 &= \boldsymbol{\omega}_i^0 \\ \dot{\theta}_B\mathbf{i_{\theta B}^0} + \dot{\phi}_B\mathbf{j_{\phi B}^0} + \dot{\psi}_B\mathbf{k_{\psi B}^0} + & \\ +\dot{\theta}_{1i}\mathbf{k_{0i}^0} + \dot{\theta}_{2i}\mathbf{j_{2i}^0} + \dot{\theta}_{3i}\mathbf{k_{4i}^0} + & \\ +\dot{\theta}_{4i}\mathbf{k_{6i}^0} + \dot{\theta}_{5i}\mathbf{k_{8i}^0} + \dot{\theta}_{6i}\mathbf{i_{10i}^0} &= \dot{\theta}_i\mathbf{i_{\theta i}^0} + \dot{\phi}_i\mathbf{j_{\phi i}^0} + \dot{\psi}_i\mathbf{k_{\psi i}^0} \end{aligned} \quad (9)$$

where $\boldsymbol{\omega}_i^0$ are angular velocity vectors corresponding to each foot.

Assuming that numerical solution of equations (4) already gave the angular position values $\theta_{1i}, \theta_{2i}, \theta_{3i}, \theta_{4i}, \theta_{5i}, \theta_{6i}$, system equation (8) - (9) is square, since they can be posed as 6 scalar equations with 6 unknown variables: $\dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$

C. Acceleration equations

Time derivative of velocity equation (6) can be expressed as follows:

$$\dot{\mathbf{v}}_B^0 + \dot{\mathbf{v}}_{0i}^0 + \dot{\mathbf{v}}_{1i}^0 + \dot{\mathbf{v}}_{2i}^0 + \dot{\mathbf{v}}_{3i}^0 + \dot{\mathbf{v}}_{4i}^0 + \dot{\mathbf{v}}_{5i}^0 + \dot{\mathbf{v}}_{6i}^0 = \dot{\mathbf{v}}_i^0$$

If one define $\mathbf{a} = \dot{\mathbf{v}}$, acceleration equation for translation is:

$$\mathbf{a_B^0} + \mathbf{a_{0i}^0} + \mathbf{a_{1i}^0} + \mathbf{a_{2i}^0} + \mathbf{a_{3i}^0} + \mathbf{a_{4i}^0} + \mathbf{a_{5i}^0} + \mathbf{a_{6i}^0} = \mathbf{a_i^0} \quad (10)$$

where acceleration vectors with respect to the inertial frame are given by:

$$\begin{aligned} \mathbf{a_B^0} &= \ddot{x}_B\mathbf{i_0} + \ddot{y}_B\mathbf{j_0} + \ddot{z}_B\mathbf{k_0} \\ \mathbf{a_{0i}^0} &= \boldsymbol{\alpha}_B^0 \times \mathbf{r_{0i}^0} + \boldsymbol{\omega}_{\mathbf{B}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{B}}^{\mathbf{0}} \times \mathbf{r_{0i}^0}\right) \\ \mathbf{a_{1i}^0} &= \boldsymbol{\alpha}_{1i}^0 \times \mathbf{r_{1i}^0} + \boldsymbol{\omega}_{\mathbf{1i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{1i}}^{\mathbf{0}} \times \mathbf{r_{1i}^0}\right) \\ \mathbf{a_{2i}^0} &= \boldsymbol{\alpha}_{2i}^0 \times \mathbf{r_{2i}^0} + \boldsymbol{\omega}_{\mathbf{2i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{2i}}^{\mathbf{0}} \times \mathbf{r_{2i}^0}\right) \\ \mathbf{a_{3i}^0} &= \boldsymbol{\alpha}_{3i}^0 \times \mathbf{r_{3i}^0} + \boldsymbol{\omega}_{\mathbf{3i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{3i}}^{\mathbf{0}} \times \mathbf{r_{3i}^0}\right) \\ \mathbf{a_{4i}^0} &= \boldsymbol{\alpha}_{4i}^0 \times \mathbf{r_{4i}^0} + \boldsymbol{\omega}_{\mathbf{4i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{4i}}^{\mathbf{0}} \times \mathbf{r_{4i}^0}\right) \\ \mathbf{a_{5i}^0} &= \boldsymbol{\alpha}_{5i}^0 \times \mathbf{r_{5i}^0} + \boldsymbol{\omega}_{\mathbf{5i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{5i}}^{\mathbf{0}} \times \mathbf{r_{5i}^0}\right) \\ \mathbf{a_{6i}^0} &= \boldsymbol{\alpha}_{6i}^0 \times \mathbf{r_{6i}^0} + \boldsymbol{\omega}_{\mathbf{6i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{6i}}^{\mathbf{0}} \times \mathbf{r_{6i}^0}\right) \\ \mathbf{a_i^0} &= \ddot{x}_i\mathbf{i_0} + \ddot{y}_i\mathbf{j_0} + \ddot{z}_i\mathbf{k_0} \end{aligned}$$

When these vectors are substituted in (10), the acceleration equations for the translation of each leg are:

$$\begin{aligned} \ddot{x}_B\mathbf{i_0} + \ddot{y}_B\mathbf{j_0} + \ddot{z}_B\mathbf{k_0} &\quad + \\ \boldsymbol{\alpha}_B^0 \times \mathbf{r_{0i}^0} + \boldsymbol{\omega}_{\mathbf{B}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{B}}^{\mathbf{0}} \times \mathbf{r_{0i}^0}\right) &\quad + \\ \boldsymbol{\alpha}_{1i}^0 \times \mathbf{r_{1i}^0} + \boldsymbol{\omega}_{\mathbf{1i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{1i}}^{\mathbf{0}} \times \mathbf{r_{1i}^0}\right) &\quad + \\ \boldsymbol{\alpha}_{2i}^0 \times \mathbf{r_{2i}^0} + \boldsymbol{\omega}_{\mathbf{2i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{2i}}^{\mathbf{0}} \times \mathbf{r_{2i}^0}\right) &\quad + \\ \boldsymbol{\alpha}_{3i}^0 \times \mathbf{r_{3i}^0} + \boldsymbol{\omega}_{\mathbf{3i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{3i}}^{\mathbf{0}} \times \mathbf{r_{3i}^0}\right) &\quad + \\ \boldsymbol{\alpha}_{4i}^0 \times \mathbf{r_{4i}^0} + \boldsymbol{\omega}_{\mathbf{4i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{4i}}^{\mathbf{0}} \times \mathbf{r_{4i}^0}\right) &\quad + \\ \boldsymbol{\alpha}_{5i}^0 \times \mathbf{r_{5i}^0} + \boldsymbol{\omega}_{\mathbf{5i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{5i}}^{\mathbf{0}} \times \mathbf{r_{5i}^0}\right) &\quad + \\ \boldsymbol{\alpha}_{6i}^0 \times \mathbf{r_{6i}^0} + \boldsymbol{\omega}_{\mathbf{6i}}^{\mathbf{0}} \times \left(\boldsymbol{\omega}_{\mathbf{6i}}^{\mathbf{0}} \times \mathbf{r_{6i}^0}\right) &= \ddot{x}_i\mathbf{i_0} + \ddot{y}_i\mathbf{j_0} + \ddot{z}_i\mathbf{k_0} \end{aligned} \quad (11)$$

where $\boldsymbol{\alpha}_B^0$ is the angular acceleration vector of the torso and $\boldsymbol{\alpha}_{ni}^0$ is the angular acceleration vector of the ni-th link.

Angular acceleration equations are obtained by computing the time derivative of (9):

$$\dot{\boldsymbol{\omega}}_{6i}^0 = \dot{\boldsymbol{\omega}}_i^0$$

$$\boldsymbol{\alpha}_{6i}^0 = \boldsymbol{\alpha}_i^0$$

$$\begin{aligned} &\ddot{\theta}_B\mathbf{i_0} + \ddot{\phi}_B\mathbf{j_{\phi_B}^0} + \ddot{}_B\mathbf{k_{\psi_B}^0} + \dot{\theta}_B\left(\mathbf{i_{\theta_B}^0} \times \boldsymbol{\omega}_{\phi_B}^0\right) \\ &+\dot{\phi}_B\left(\mathbf{j_{\phi_B}^0} \times \boldsymbol{\omega}_{\psi_B}^0\right) + \dot{\psi}_B\left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k_{\psi_B}^0}\right) \\ &+\ddot{\theta}_{1i}\mathbf{k_{0i}^0} + \dot{\theta}_{1i}\left(\boldsymbol{\omega}_B^0 \times \mathbf{k_{0i}^0}\right) + \ddot{\theta}_{2i}\mathbf{j_{2i}^0} + \dot{\theta}_{2i}\left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{j_{2i}^0}\right) \\ &+\ddot{\theta}_{3i}\mathbf{k_{4i}^0} + \dot{\theta}_{3i}\left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{k_{4i}^0}\right) + \ddot{\theta}_{4i}\mathbf{k_{6i}^0} + \dot{\theta}_{4i}\left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{k_{6i}^0}\right) \\ &+\ddot{\theta}_{5i}\mathbf{k_{8i}^0} + \dot{\theta}_{5i}\left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{k_{8i}^0}\right) + \ddot{\theta}_{6i}\mathbf{i_{10i}^0} \\ &+\dot{\theta}_{6i}\left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{i_{10i}^0}\right) \quad = \quad \ddot{\theta}_i\mathbf{i_0} + \ddot{\phi}_i\mathbf{j_{\phi_i}^0} + \ddot{}_i\mathbf{k_{\psi_i}^0} + \\ &+\dot{\theta}_i\left(\mathbf{i_{\theta_i}^0} \times \boldsymbol{\omega}_{\phi_i}^0\right) + \dot{\phi}_i\left(\mathbf{j_{\phi_i}^0} \times \boldsymbol{\omega}_{\psi_i}^0\right) + \dot{\psi}_i\left(\boldsymbol{\omega}_{\theta_i}^0 \times \mathbf{k_{\psi_i}^0}\right) \end{aligned} \quad (12)$$

Assuming that numerical solutions of equations of position and velocity have been already computed, that is, numerical values of $\theta_{1i}, \theta_{2i}, \ldots, \theta_{6i}$ and $\dot{\theta}_{1i}, \dot{\theta}_{2i}, \ldots, \dot{\theta}_{6i}$, system equation (11) - (12) can be solved since they are 6 scalar equations with 6 unknown variables: $\ddot{\theta}_{1i}, \ddot{\theta}_{2i}, \ddot{\theta}_{3i}, \ddot{\theta}_{4i}, \ddot{\theta}_{5i}, \ddot{\theta}_{6i}$

III. Computation of the ZMP

The vectors that describe the position of the center of gravity of the torso and of the ni-th link, $\mathbf{r^0_{GB}}$ and $\mathbf{r^0_{Gni}}$ respectively , are defined as

$$
\begin{aligned}
\mathbf{r^0_{GB}} &= \mathbf{R^0_B r^B_{GB}} & \mathbf{r^0_{G1i}} &= \mathbf{R^0_{1i} r^{1i}_{G1i}}\\
\mathbf{r^0_{G2i}} &= \mathbf{R^0_{3i} r^{3i}_{G2i}} & \mathbf{r^0_{G3i}} &= \mathbf{R^0_{5i} r^{5i}_{G3i}}\\
\mathbf{r^0_{G4i}} &= \mathbf{R^0_{7i} r^{7i}_{G4i}} & \mathbf{r^0_{G5i}} &= \mathbf{R^0_{9i} r^{9i}_{G5i}}\\
\mathbf{r^0_{G6i}} &= \mathbf{R^0_{11i} r^{11i}_{G6i}}
\end{aligned}
$$

where $\mathbf{r^B_{GB}}$ and $\mathbf{r^{mi}_{Gni}}$ are numerically given in appendix B.

Using these vectors and those obtained from the solution for velocity, the position of centers of gravity of the torso and the ni-link,$\mathbf{b^0_B}$ and $\mathbf{b^0_{ni}}$ respectively, with respect to the inertial frame are given by:

$$
\begin{aligned}
\mathbf{b^0_B} &= \mathbf{r^0_B} + \mathbf{r^0_{GB}}\\
\mathbf{b^0_{1i}} &= \mathbf{r^0_B} + \mathbf{r^0_{0i}} + \mathbf{r^0_{G1i}}\\
\mathbf{b^0_{2i}} &= \mathbf{r^0_B} + \mathbf{r^0_{0i}} + \mathbf{r^0_{1i}} + \mathbf{r^0_{G2i}}\\
\mathbf{b^0_{3i}} &= \mathbf{r^0_B} + \mathbf{r^0_{0i}} + \mathbf{r^0_{1i}} + \mathbf{r^0_{2i}} + \mathbf{r^0_{G3i}}\\
\mathbf{b^0_{4i}} &= \mathbf{r^0_B} + \mathbf{r^0_{0i}} + \mathbf{r^0_{1i}} + \mathbf{r^0_{2i}} + \mathbf{r^0_{3i}} + \mathbf{r^0_{G4i}}\\
\mathbf{b^0_{5i}} &= \mathbf{r^0_B} + \mathbf{r^0_{0i}} + \mathbf{r^0_{1i}} + \mathbf{r^0_{2i}} + \mathbf{r^0_{3i}} + \mathbf{r^0_{4i}} + \mathbf{r^0_{G5i}}\\
\mathbf{b^0_{6i}} &= \mathbf{r^0_B} + \mathbf{r^0_{0i}} + \mathbf{r^0_{1i}} + \mathbf{r^0_{2i}} + \mathbf{r^0_{3i}} + \mathbf{r^0_{4i}} + \mathbf{r^0_{5i}} + \mathbf{r^0_{G6i}}
\end{aligned}
$$

Vectors $\mathbf{b^0_B}$, $\mathbf{b^0_{ni}}$ are illustrated in Figure 9.

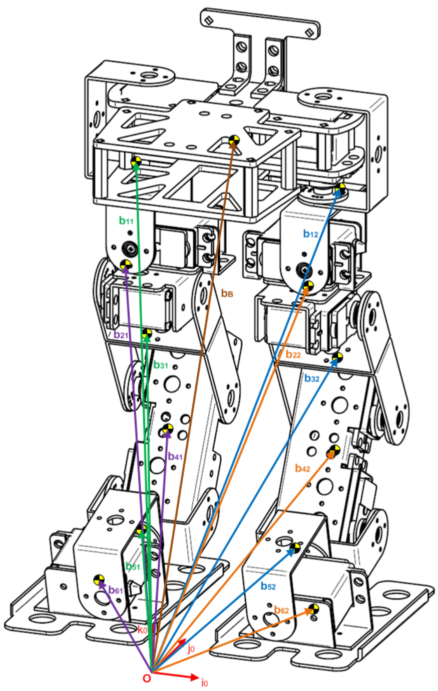

Fig. 9. Vectors that define the gravity center of each link and the torso.

Matrices $\mathbf{I^0_B}$, $\mathbf{I^0_{ni}}$ represent the inertia moments of the links with respect to the inertial frame:

$$
\begin{aligned}
\mathbf{I^0_B} &= \mathbf{R^0_B I^B_B}\left(\mathbf{R^0_B}\right)^{\mathbf{T}} & \mathbf{I^0_{1i}} &= \mathbf{R^0_{1i} I^{1i}_{1i}}\left(\mathbf{R^0_{1i}}\right)^{\mathbf{T}}\\
\mathbf{I^0_{2i}} &= \mathbf{R^0_{3i} I^{3i}_{2i}}\left(\mathbf{R^0_{3i}}\right)^{\mathbf{T}} & \mathbf{I^0_{3i}} &= \mathbf{R^0_{5i} I^{5i}_{3i}}\left(\mathbf{R^0_{5i}}\right)^{\mathbf{T}}\\
\mathbf{I^0_{4i}} &= \mathbf{R^0_{7i} I^{7i}_{4i}}\left(\mathbf{R^0_{7i}}\right)^{\mathbf{T}} & \mathbf{I^0_{5i}} &= \mathbf{R^0_{9i} I^{9i}_{5i}}\left(\mathbf{R^0_{9i}}\right)^{\mathbf{T}}\\
\mathbf{I^0_{6i}} &= \mathbf{R^0_{11i} I^{11i}_{6i}}\left(\mathbf{R^0_{11i}}\right)^{\mathbf{T}}
\end{aligned}
$$

According to the method reported in [2], one can obtain the center of mass vector $\mathbf{b^0_T}$, its total linear momentum $\mathcal{P}$ and its total angular momentum $\mathcal{L}$ with respect to the inertial frame as follows:

$$
\begin{aligned}
\mathbf{b^0_T} &= \frac{m_B}{m_T}\mathbf{b^0_B} + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{m_{ni}}{m_T}\mathbf{b^0_{ni}}\right)\\
\mathcal{P} &= m_B\dot{\mathbf{b}}^0_{\mathbf{B}} + \sum_{i=1}^{2}\left(\sum_{n=1}^{6} m_{ni}\dot{\mathbf{b}}^0_{\mathbf{ni}}\right)\\
\mathcal{L} &= \mathbf{b^0_B}\times\left(m_B\dot{\mathbf{b}}^0_{\mathbf{B}}\right) + \mathbf{I^0_B}\boldsymbol{\omega}^0_{\mathbf{B}} +\\
&\quad + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\mathbf{b^0_{ni}}\times\left(m_{ni}\dot{\mathbf{b}}^0_{\mathbf{ni}}\right) + \mathbf{I^0_{ni}}\boldsymbol{\omega}^0_{\mathbf{ni}}\right)\right)
\end{aligned}
$$

One can refer to the elements of the previous column vectors as: $\mathcal{P} = \begin{bmatrix} \mathcal{P}_x & \mathcal{P}_y & \mathcal{P}_z \end{bmatrix}^T$, $\mathcal{L} = \begin{bmatrix} \mathcal{L}_x & \mathcal{L}_y & \mathcal{L}_z \end{bmatrix}^T$, $\mathbf{b^0_T} = \begin{bmatrix} b_{Tx} & b_{Ty} & b_{Tz} \end{bmatrix}^T$.

The position vector associated to the Zero Moment Point $\mathbf{p_{ZMP}}$ with respect to the inertial frame is represented as $\mathbf{p_{ZMP}} = \begin{bmatrix} ZMP_x & ZMP_y & ZMP_z \end{bmatrix}^{\mathbf{T}}$, where:

$$
\begin{aligned}
ZMP_x &= \frac{m_T g b_{Tx} + ZMP_z\dot{\mathcal{P}}_x - \dot{\mathcal{L}}_y}{m_T g + \dot{\mathcal{P}}_z}\\
ZMP_y &= \frac{m_T g b_{Ty} + ZMP_z\dot{\mathcal{P}}_y + \dot{\mathcal{L}}_x}{m_T g + \dot{\mathcal{P}}_z}
\end{aligned}
\tag{13}
$$

and ZMP_z is the height of the floor plane with respect to the inertial frame, which in this case is zero.

IV. Numerical simulations

The walking pattern was chosen for these simulations as simple as possible, that is, considering that: (a) the feet are always parallel to the floor-plane, (b) point $\mathbf{p_B}$ moves in the sagittal plane during all the walk, (c) coordinate z_B is always constant. Under these restrictions, positions of points $\mathbf{p_B}, \mathbf{p_i}$ and the orientations of local frames $(\mathbf{p_B}; \mathbf{i_B}, \mathbf{j_B}, \mathbf{k_B})$, $(\mathbf{p_i}; \mathbf{i_i}, \mathbf{j_i}, \mathbf{k_i})$ with respect to the inertial frame $(\mathbf{O}; \mathbf{i_0}, \mathbf{j_0}, \mathbf{k_0})$ as function of time t are geometrically restricted by: $x_B(t) = 0$, $z_B(t) = 234.37[mm]$, $x_1(t) = x_2(t) = 44.54[mm]$, $\theta_B(t) = \theta_1(t) = \theta_2(t) = 0$, $\phi_B(t) = \phi_1(t) = \phi_2(t) = 0$, $\psi_B(t) = \psi_1(t) = \psi_2(t) = 0$.

Functions $y_B(t)$, $y_1(t)$, $z_1(t)$, $y_2(t)$, $z_2(t)$ are then freely assigned.

TABLE I. Geometric parameters. All lengths are in milimeters, that is $1 \times 10^{-3}[m]$

$x_{0i} = 44.54$	$z_{1i} = 54.58$	$x_{2i} = 29.08$
$x_{3i} = 74.60$	$x_{4i} = 57.29$	$x_{5i} = 54.97$
$y_{6i} = 23.06$	$l_1 = 75.24$	$l_2 = 127$

TABLE II. Angles β_{ni} in degrees, [°]

$\beta_{11} = 180$	$\beta_{21} = 90$	$\beta_{31} = 330$	$\beta_{41} = 60$
$\beta_{51} = 45$	$\beta_{61} = 180$	$\beta_{12} = 0$	$\beta_{22} = 90$
$\beta_{32} = 30$	$\beta_{42} = 300$	$\beta_{52} = 315$	$\beta_{62} = 0$
$\beta_7 = 75$		$\beta_8 = 270$	

Based on the physical dimensions of the robot, a CAD model of the robot was generated using the specialized software Solid Edge Academic© V16, where the geometric parameters of Tables I and II were introduced as well as the position vectors to draw the feet structures (defined in Appendix A).

With the help of this CAD model, instantaneous values of coordinates y_B, y_1, z_1, y_2, z_2 were obtained and then used to generate second-degree splines that are taken as reference trajectories for the movement of the robot during $40[s]$. The generated splines are:

$$y_B(t) = \begin{cases} 2.23t + 38.5; & 0 \le t < 15 \\ 0.03t^2 + 1.23t + 46; & 15 \le t < 20 \\ -0.24t + 12.26t - 64.25; & 20 \le t < 25 \\ 90.76; & 25 \le t < 40 \end{cases}$$

$$y_1(t) = \begin{cases} 0; & 0 \le t < 20 \\ 1.93t - 38.67; & 20 \le t < 35 \\ 1.13t^2 - 77.4t + 1349.67; & 35 \le t < 38 \\ -2.55t^2 + 202.66t - 3971.47; & 38 \le t < 40 \end{cases}$$

$$z_1(t) = \begin{cases} 0; & 0 \le t < 20 \\ 4t - 80; & 20 \le t < 25 \\ -\frac{2}{5}t^2 + 24t - 330; & 25 \le t < 35 \\ -4t + 160; & 35 \le t \le 40 \end{cases}$$

$$y_2(t) = \begin{cases} 1.93t; & 0 \le t < 15 \\ 1.13t^2 - 32.07t + 255; & 15 \le t < 18 \\ -2.56t^2 + 100.59t - 938.94; & 18 \le t < 20 \\ 52.26; & 20 \le t < 40 \end{cases}$$

$$z_2(t) = \begin{cases} 4t; & 0 \le t < 5 \\ -\frac{2}{5}t^2 + 8t - 10; & 5 \le t < 15 \\ -4t + 80; & 15 \le t \le 20 \\ 0; & 20 \le t < 40 \end{cases}$$

These piecewise functions as well as its analytic time derivatives were used to solve position, velocity and acceleration equations. Results are presented in Figures 10 to 13.

Figure 10 presents the walking cycle using the simplified model of the robot, where the fulfillment of geometric restrictions can be verified. A small cylinder was drew at the bottom in order to have a reference point to appreciate the translation of the robot. In the same figure the position of the ZMP is drawn. It can be seen that it is outside the support polygon when the robot is in the single support phases.

Figures 11, 12 and 13 present the evolution in time of position, velocity and acceleration of the rotational joints of the robot, respectively (joint angles that remain at cero for all time are not presented for obvious reasons).

These results were introduced as a sequence to move the servomotors of the robot in order to verifiy the theoretical walking cycle to the real system. In order to verify the generated movement, a videoclip can be seen in http://www.youtube.com/watch?v=mpRDsCrwvwk.

V. Conclusions

The obtained kinematic model is adequate since generates satisfactory results for the analysis of position, velocity and acceleration of the biped robot. This model has been already used to obtain the dynamic model by the Euler-Lagrange approach.

Simulation results correspond to one of several possible walking cycles. From the implementation of the generated trajectories of the joints based on the second-degree splines and the obtained positions of the ZMP, it can be stated that the gait cycle is not dynamically stable (as can be verified in the videoclip). We assume that this was due to the resticted way in which the positions of the torso and feet were proposed.

The next step in our research is to generate new spatial configurations for the biped based on desired trajectories for the ZMP.

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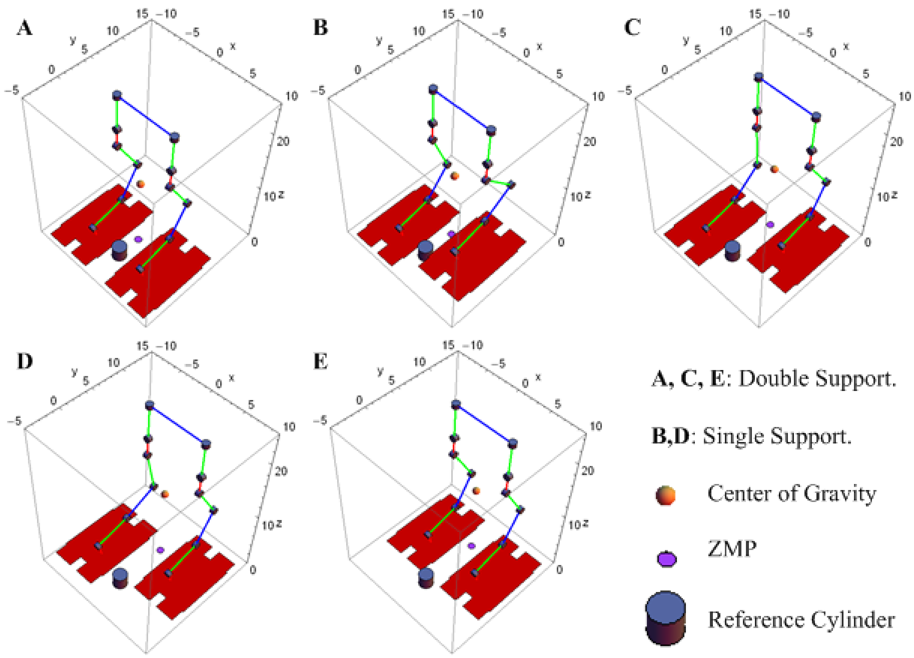

Fig. 10. Simulation results: walking cycle.

[8] Olli Haavisto and Heikki Hyotyniemi. Simulation tool of a biped walking robot model. Technical report, Helsinki University of Technology Control Engineering Laboratory, 2004.
[9] John J. Craig. Introduction to Robotics: Mechanics and Control. Prentice Hall, EUA, 2004.

Appendix

I. Vectorial representation of feet

In Tables III and IV the position vectors that define the base of the feet are given.

TABLE III. All units are in milimeters, that is $1 \times 10^{-3}[m]$

Left foot:	
$\mathbf{r^1_{a1}} = -\mathbf{2.45i_1} + \mathbf{64.34j_1}$	$\mathbf{r^1_{b1}} = -\mathbf{2.45i_1} + \mathbf{83.33j_1}$
$\mathbf{r^1_{c1}} = \mathbf{26.18i_1} + \mathbf{83.33j_1}$	$\mathbf{r^1_{d1}} = \mathbf{26.18i_1} + \mathbf{64.34j_1}$
$\mathbf{r^1_{e1}} = \mathbf{28.18i_1} + \mathbf{64.81j_1}$	$\mathbf{r^1_{f1}} = \mathbf{28.18i_1} - \mathbf{24.20j_1}$
$\mathbf{r^1_{g1}} = \mathbf{26.18i_1} - \mathbf{24.70j_1}$	$\mathbf{r^1_{h1}} = \mathbf{26.18i_1} - \mathbf{43.70j_1}$
$\mathbf{r^1_{i1}} = -\mathbf{2.45i_1} - \mathbf{43.70j_1}$	$\mathbf{r^1_{j1}} = -\mathbf{2.45i_1} - \mathbf{24.7j_1}$
$\mathbf{r^1_{k1}} = -\mathbf{15.12i_1} - \mathbf{24.7j_1}$	$\mathbf{r^1_{l1}} = -\mathbf{15.12i_1} - \mathbf{43.70j_1}$
$\mathbf{r^1_{m1}} = -\mathbf{44.33i_1} - \mathbf{43.70j_1}$	$\mathbf{r^1_{n1}} = -\mathbf{44.33i_1} - \mathbf{24.70j_1}$
$\mathbf{r^1_{o1}} = -\mathbf{46.33i_1} - \mathbf{24.20j_1}$	$\mathbf{r^1_{p1}} = -\mathbf{46.33i_1} + \mathbf{64.81j_1}$
$\mathbf{r^1_{q1}} = -\mathbf{44.33i_1} + \mathbf{64.34j_1}$	$\mathbf{r^1_{r1}} = -\mathbf{44.33i_1} + \mathbf{83.33j_1}$
$\mathbf{r^1_{s1}} = -\mathbf{15.12i_1} + \mathbf{83.33j_1}$	$\mathbf{r^1_{t1}} = -\mathbf{15.12i_1} + \mathbf{64.34j_1}$

II. Vectors that define the position of the gravity centers

Torso (in milimeters, that is $1 \times 10^{-3}[m]$):

$$\mathbf{r^B_{GB}} = -\mathbf{7.69j_B} - \mathbf{0.62k_B}$$

TABLE IV. All units are in milimeters, that is $1 \times 10^{-3}[m]$

Right foot:	
$\mathbf{r^2_{a2}} = \mathbf{15.70i_2} + \mathbf{64.34j_2}$	$\mathbf{r^2_{b2}} = \mathbf{15.70i_2} + \mathbf{83.33j_2}$
$\mathbf{r^2_{c2}} = \mathbf{44.33i_2} + \mathbf{83.33j_2}$	$\mathbf{r^2_{d2}} = \mathbf{44.33i_2} + \mathbf{64.34j_2}$
$\mathbf{r^2_{e2}} = \mathbf{46.33i_2} + \mathbf{64.81j_2}$	$\mathbf{r^2_{f2}} = \mathbf{46.33i_2} - \mathbf{24.20j_2}$
$\mathbf{r^2_{g2}} = \mathbf{44.33i_2} - \mathbf{24.70j_2}$	$\mathbf{r^2_{h2}} = \mathbf{44.33i_2} - \mathbf{43.70j_2}$
$\mathbf{r^2_{i2}} = \mathbf{15.70i_2} - \mathbf{43.70j_2}$	$\mathbf{r^2_{j2}} = \mathbf{15.70i_2} - \mathbf{24.70j_2}$
$\mathbf{r^2_{k2}} = \mathbf{3.03i_2} - \mathbf{24.70j_2}$	$\mathbf{r^2_{l2}} = \mathbf{3.03i_2} - \mathbf{43.70j_2}$
$\mathbf{r^2_{m2}} = -\mathbf{26.18i_2} - \mathbf{43.70j_2}$	$\mathbf{r^2_{n2}} = -\mathbf{26.18i_2} - \mathbf{24.70j_2}$
$\mathbf{r^2_{o2}} = -\mathbf{28.18i_2} - \mathbf{24.20j_2}$	$\mathbf{r^2_{p2}} = -\mathbf{28.18i_2} + \mathbf{64.81j_2}$
$\mathbf{r^2_{q2}} = -\mathbf{26.18i_2} + \mathbf{64.34j_2}$	$\mathbf{r^2_{r2}} = -\mathbf{26.18i_2} + \mathbf{83.33j_2}$
$\mathbf{r^2_{s2}} = \mathbf{3.03i_2} + \mathbf{83.33j_2}$	$\mathbf{r^2_{t2}} = \mathbf{3.03i_2} + \mathbf{64.34j_2}$

In Table V the position vectors of the centers of gravity of the feet are given.

III. Inertia moments of the links

All units are in $1 \times 10^{-6}\left[kg \cdot m^2\right]$.
Torso:

$$\mathbf{I^B_B} = \begin{bmatrix} 152.08 & -0.09 & -0.02 \\ -0.09 & 214.21 & 1.63 \\ -0.02 & 1.63 & 306.38 \end{bmatrix}$$

Left foot:

$$\mathbf{I^{11}_{11}} = \begin{bmatrix} 29.40 & 0.04 & -0.04 \\ 0.04 & 29.04 & -0.09 \\ -0.04 & -0.09 & 11.63 \end{bmatrix}, \ \mathbf{I^{31}_{21}} = \begin{bmatrix} 38.74 & 9.88 & -5.06 \\ 9.88 & 63.10 & -0.27 \\ -5.06 & -0.27 & 61.51 \end{bmatrix}$$

$$\mathbf{I^{51}_{31}} = \begin{bmatrix} 14.47 & 0.08 & -1.09 \\ 0.08 & 57.65 & -0.00 \\ -1.09 & -0.00 & 45.90 \end{bmatrix}, \ \mathbf{I^{71}_{41}} = \begin{bmatrix} 21.97 & 4.37 & -6.21 \\ 4.37 & 155.35 & -1.82 \\ -6.21 & -1.82 & 144.59 \end{bmatrix}$$

$$\mathbf{I^{91}_{51}} = \begin{bmatrix} 11.62 & 10.53 & -0.98 \\ 10.53 & 50.49 & -0.23 \\ -0.98 & -0.23 & 50.05 \end{bmatrix}, \ \mathbf{I^{111}_{61}} = \begin{bmatrix} 63.40 & -26.31 & 6.48 \\ -26.31 & 93.08 & -8.23 \\ 6.48 & -8.23 & 86.73 \end{bmatrix}$$

TABLE V. All units are in milimeters, that is $1 \times 10^{-3}[m]$

Left foot:

$$\mathbf{r}_{G11}^{11} = 8.49\mathbf{i}_{11} + 0.22\mathbf{j}_{11} - 18.77\mathbf{k}_{11}$$
$$\mathbf{r}_{G21}^{31} = 14.48\mathbf{i}_{31} + 2.60\mathbf{j}_{31} - 6.24\mathbf{k}_{31}$$
$$\mathbf{r}_{G31}^{51} = 34.33\mathbf{i}_{51} + 0.08\mathbf{j}_{51} - 1.21\mathbf{k}_{51}$$
$$\mathbf{r}_{G41}^{71} = 19.52\mathbf{i}_{71} + 2.13\mathbf{j}_{71} - 3.44\mathbf{k}_{71}$$
$$\mathbf{r}_{G51}^{91} = 35.49\mathbf{i}_{91} + 8.37\mathbf{j}_{91} - 1.21\mathbf{k}_{91}$$
$$\mathbf{r}_{G61}^{111} = -9.58\mathbf{i}_{111} + 11.39\mathbf{j}_{111} - 7.92\mathbf{k}_{111}$$

Right foot:

$$\mathbf{r}_{G12}^{12} = 8.49\mathbf{i}_{12} + 0.22\mathbf{j}_{12} - 18.77\mathbf{k}_{12}$$
$$\mathbf{r}_{G22}^{32} = 14.48\mathbf{i}_{32} - 2.60\mathbf{j}_{32} - 6.24\mathbf{k}_{32}$$
$$\mathbf{r}_{G32}^{52} = 34.33\mathbf{i}_{52} - 0.08\mathbf{j}_{52} - 1.21\mathbf{k}_{52}$$
$$\mathbf{r}_{G42}^{72} = 19.52\mathbf{i}_{72} - 2.13\mathbf{j}_{72} - 3.44\mathbf{k}_{72}$$
$$\mathbf{r}_{G52}^{92} = 35.49\mathbf{i}_{92} - 8.37\mathbf{j}_{92} - 1.21\mathbf{k}_{92}$$
$$\mathbf{r}_{G62}^{112} = -9.58\mathbf{i}_{112} + 11.39\mathbf{j}_{112} + 7.92\mathbf{k}_{112}$$

Right foot:

$$\mathbf{I}_{12}^{12} = \begin{bmatrix} 29.40 & 0.04 & -0.04 \\ 0.04 & 29.04 & -0.09 \\ -0.04 & -0.09 & 11.63 \end{bmatrix}, \; \mathbf{I}_{22}^{32} = \begin{bmatrix} 38.74 & -9.88 & -5.06 \\ -9.88 & 63.10 & 0.27 \\ -5.06 & 0.27 & 61.51 \end{bmatrix}$$

$$\mathbf{I}_{32}^{52} = \begin{bmatrix} 14.47 & -0.08 & -1.09 \\ -0.08 & 57.65 & 0.00 \\ -1.09 & 0.00 & 45.90 \end{bmatrix}, \; \mathbf{I}_{42}^{72} = \begin{bmatrix} 21.97 & -4.37 & -6.21 \\ -4.37 & 155.35 & 1.82 \\ -6.21 & 1.82 & 144.59 \end{bmatrix}$$

$$\mathbf{I}_{52}^{92} = \begin{bmatrix} 11.62 & -10.53 & -0.98 \\ -10.53 & 50.49 & 0.23 \\ -0.98 & 0.23 & 50.05 \end{bmatrix}, \; \mathbf{I}_{62}^{112} = \begin{bmatrix} 63.40 & -26.31 & -6.48 \\ -26.31 & 93.08 & 8.23 \\ -6.48 & 8.23 & 86.73 \end{bmatrix}$$

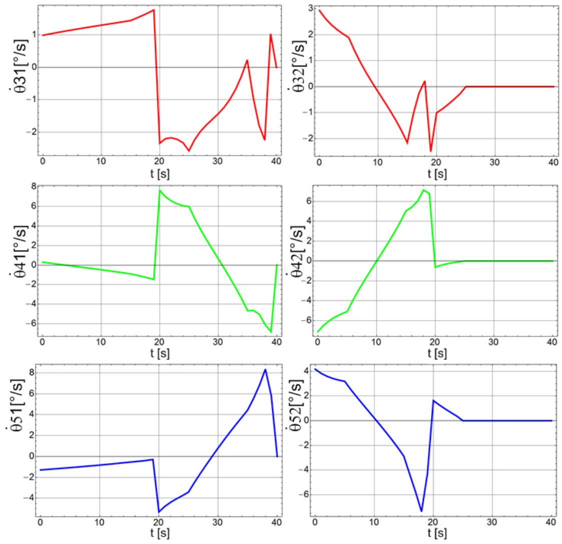

Fig. 12. Simulation results: angular velocities

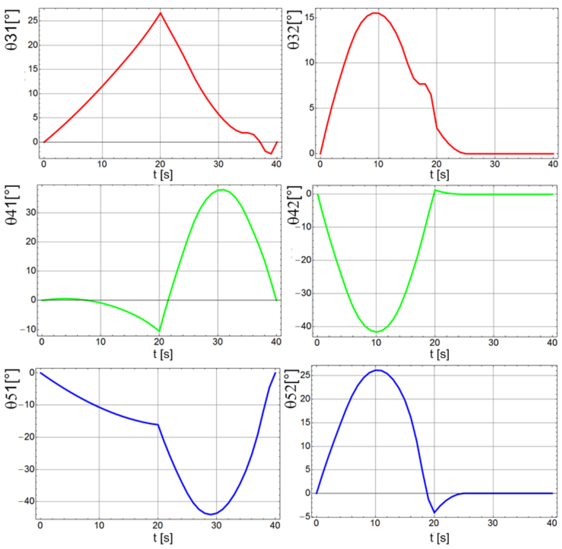

Fig. 11. Simulation results: angular positions

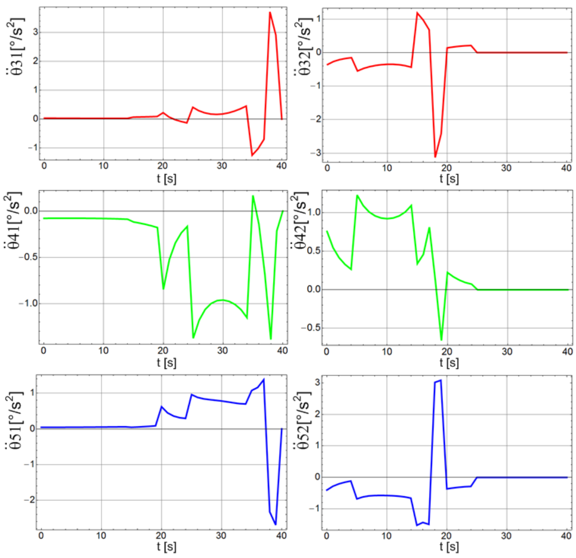

Fig. 13. Simulation results: angular accelerations