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A MATHEMATICAL MODEL FOR THE DYNAMICS OF HUMAN LOCOMOTION*

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Abstract – The problem of developing a model to describe and study human gait was undertaken using Lagrangian mechanics. The approach is that of the initial value problem, i.e. starting with the initial conditions which are the limb angles and velocities, and the system inputs which are the applied joint moments, the system response is found. Seven segments are used to model the human body with complete three segment lower limbs, and the head arms and trunk included as one segment. There are six joints, two each at the hips, knees, and ankles.

To show the behavior of the developed model a set of initial conditions and moment histories were obtained from measurements in a gait laboratory. Using these data with minor modifications, it is shown that the model progresses through the normal walking cycle, or with minor perturbations, atypical gait patterns can be demonstrated.

It is hoped that this model will provide further insight into human gait. For example, alterations of the time histories of the joint moments can be used to predict how certain muscle groups affect gait. Thus the teaching of the complexities of gait will be facilitated. Ultimately, predictions concerning the results of therapy and surgery might be made using this model as a diagnostic tool.

The emphasis in this paper is on the development of the mathematical model and not on numerical results.

INTRODUCTION

One of the major drawbacks of the study of human movement is the fact that we are forced to look at the output of the system and thereby deduce the cause of the movement. As a result, link segment biomechanical analyses have evolved to enable a prediction of the joint reaction forces and muscle moments for any particular movement. When one considers the detailed interaction of internal and external forces and the energy exchanges within and between segments it is a small wonder that one can identify diagnostic patterns. One is hard pressed for answers to quite fundamental questions:

1. In normal gait, what happens when certain muscle activity is altered, i.e. strengthened or reduced?
2. In the training of an athlete, or the therapy of the physically disabled, what changes to the present pattern could be recommended?
3. Is the movement we see the most optimal from an energy or strength point of view?
4. In the validation of certain theories of motor learning or motor control, what patterns are physically realizable?

Relatively little research has been directed at modeling the dynamics of human locomotion and most were aimed at some form of optimization. However, in all models presented there were severe constraints. Bresler *et al.* (1951), Beckett and Chang (1968) and Townsend and Seireg (1972) all assumed sinusoidal trajectories in their analyses. By ignoring the higher harmonics, especially in the velocities associated with the important kinetic energy components, they have introduced considerable error. Also, by forcing the model to travel a predetermined trajectory they have constrained a wide range of possible muscle moments and have altered the problem from one of pure synthesis (forward solution) to one of curve fitting the limb displacement histories (inverse solution). In this same manner Chao and Rim (1973) used an optimization technique to find the applied joint moments by iteratively varying these until the theoretical limb displacements fit those measured in the laboratory. This was done for one leg during the gait stride. In one case (Townsend and Seireg, 1972) the model had massless extendable legs which could not account for the dominant energy changes that take place in the lower limbs (Ralston and Lukin, 1969; Winter *et al.*, 1976).

None of these mathematical models are capable of investigator interaction expected in pure synthesis which would predict the trajectories resulting from any pattern of joint muscle moments. Such a model should have no trajectory constraints and even permit undesired patterns such as hyperextension of the knee or collapse of the leg.

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PROPOSED APPROACH

The proposed model and approach is that of the initial value problem, i.e. starting with the initial conditions which are the limb angles and velocities and system inputs which are the joint moments, the system response is found. Sometimes this is referred to as the direct dynamic problem (DDP). Much of the effort related to gait studies has been directed to solving for the forces and torques applied to the system based on motion information obtained from laboratory measurements, and is referred to as the inverse dynamic problem (IDP). The IDP is easier to solve because it can be analysed one segment at a time; whereas, the DDP is more complex because the action of all segments must be considered simultaneously.

The human body is modeled by seven segments with complete three segment lower limbs, and the head arms and trunk (HAT) included as one segment. There are six joints, two each at the hips, knees and ankles. No constraints regarding the trajectories of any of the segments have been assumed.

With the proposed model the investigator can alter the time histories of net joint moments and examine the results. Ultimately, a clinician, for example, may wish to predict the results of surgery or therapy for a patient, and thereby more objectively plan the management of his patient.

MATHEMATICAL MODEL OF THE TOTAL BODY IN GAIT

The link segment model used to represent the human body is shown in Fig. 1. For convenience, it is assumed that, initially, the right foot is forward with the right heel striking the floor. In Fig. 1(b) the joint moments are denoted by subscripted M's, with the first subscript indicating the side of the body, i.e. right or left, and the second subscript indicating the joint location, i.e. ankle, knee, or hip. The seven link segments representing the human body are assumed to move in the sagittal plane. In addition, the head, arms, and trunk (HAT) are represented by one segment, a reasonable assumption when the arms do not swing excessively (Cavagna *et al.*, 1977; Dean, 1965). Each of the feet are represented by a triangle of appropriate shape, as shown in Fig. 1(a). The forward point of the triangle was placed *ca.* 2.5 cm in front of the metatarsal joint to partially compensate for the toe action during the later stages of push-off. The link segment model just described is deemed to provide a good compromise between complexity and the accurate representation of the real situation. Additional segments rapidly increase the complexity of the resulting equations of motion because of the complex coupling among all the segments. However, less than seven segments greatly reduces the accuracy of the model.

The different phases of walking require two basically different mathematical representations. The different phases of walking are:

- Phase 1: From right-heel-strike (RHS) to left-toe-off (LTO). Here, the right heel and left toe are pivoting on the floor.
- Phase 2: From LTO to right-foot-flat (RFF). Only the right heel is pivoting on the floor.
- Phase 3: During RFF. Here, the right foot is not moving, and the remaining six segments are pivoting on the right ankle.
- Phase 4: From RHO to left heel strike. Only the right toe is pivoting on the floor.

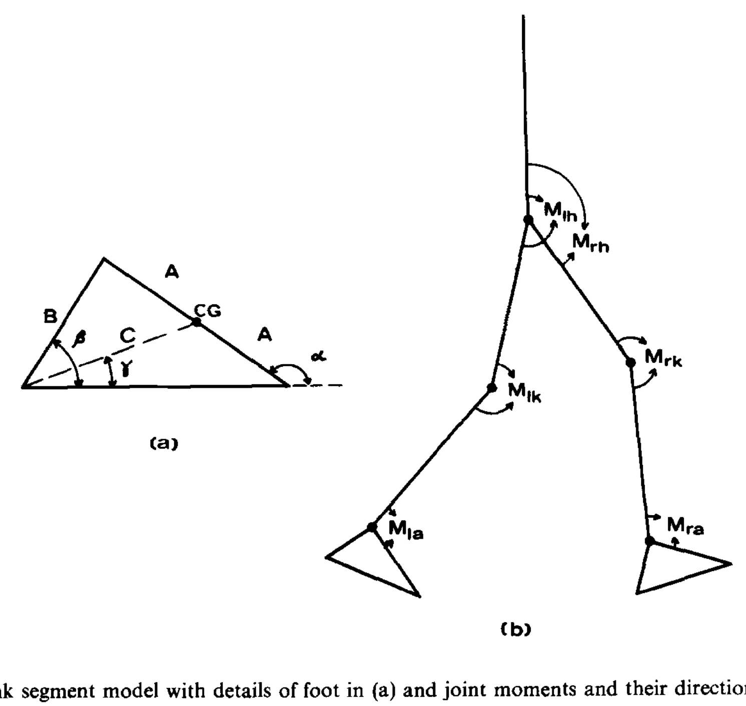

Fig. 1. Link segment model with details of foot in (a) and joint moments and their directions in (b).

The first mathematical model is required during Phase 1, where the six variables describing the two lower limbs are not independent. The constraint equations describing this situation are very complex; therefore, as suggested by Hemami *et al.* (1975), the so-called "hard constraint" is replaced by a "soft constraint". That is, the right thigh is connected to the pivot between the left thigh and HAT by a stiff spring and a damper. This, in effect, removes the constraint, and yet, the two thighs and HAT are held together at the pivot by the stiff spring. Initially, the spring-damper combination was placed at the left toe. However, this did not work well because the mass of human foot is just over a kilogramm, and, by necessity, the constraining spring must be stiff. This caused oscillations in the left foot which were very difficult to damp out. The second set of equations of motion, is required during the swing phase, Phases 2, 3 and 4, where, only small variations are required to account for the differences.

The development of the equations of motion involved several different approaches, including the use of various coordinate systems along with the application of Newtonian and Lagrangian mechanics. The temptation to use Cartesian coordinates for describing the position of the various body limbs, soon revealed that there are more constraint equations than equations of motion. Further experimenting with various coordinates resulted in a final choice of limb angles as the variables, as shown in Fig. 2. The main reason for this choice is that the limb angles have the fewest constraints associated with them. Now, Lagrangian mechanics provides the most direct approach for obtaining equations of motion.

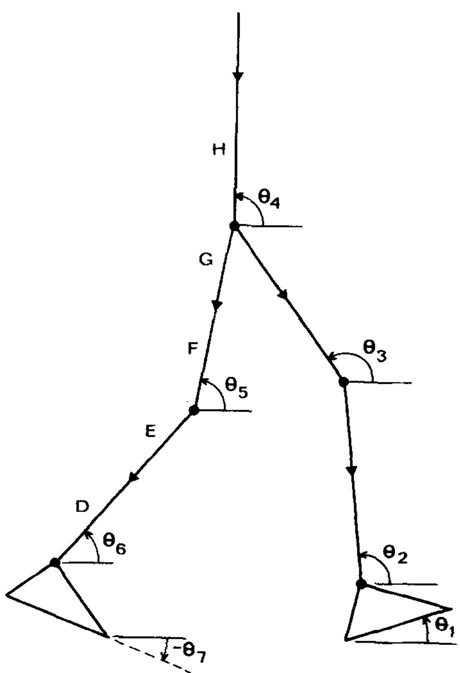

Fig. 2. Link segment model showing limb angles and limb length variables (small triangles indicate centers of gravity).

The Lagrangian mechanics approach requires the formation of the Lagrangian L, given by,

$$L = T - V, \tag{1}$$

where T and V are the kinetic and potential energies of the system, respectively, written in terms of the independent variables, the θ_i. The resulting equations of motion are given by,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = Q_i, \quad i = 1, \ldots, 7, \tag{2}$$

where the Q_i are the virtual work expressions which involve the applied joint moments and the effect of the damper in the hip during Phase 1 of the walking cycle. Note, the effect of the spring, which is across the damper is included in the potential energy term V. The details of the formation of the Lagrangian L in equation (1) and the equations of motion shown in equation (2) are not given because the procedure is straight forward and the intermediate expressions are long and awkward, consequently, of questionable interest. Instead, only the final equations of motion will be given.

The equations of motion for the double support case (Phase 1), arranged such that only the second derivatives are kept on the left hand side, has the following form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} & a_{47} \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} & a_{57} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} & a_{67} \\ 0 & 0 & 0 & a_{74} & a_{75} & a_{76} & a_{77} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \\ \ddot{\theta}_6 \\ \ddot{\theta}_7 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix}, \tag{3}$$

or,

$$[a_{ij}]\ddot{\theta}_i = b_i.$$

The terms a_{ij} involve anthropometric constants and the segment angles themselves, rendering the equations of motion nonlinear. The form of a_{ij} is,

$$a_{ij} = c_{ij}\cos(\theta_i - \theta_j). \tag{4}$$

Since the c_{ij} are symmetrical, i.e. c_{ij} equals c_{ji}, this makes the a_{ij} symmetrical, also.

On the right hand side of equation (3), the b_i are more involved and contain moments caused by the spring-damper combination, moments caused by gravity, the applied joint moments, and terms involving the square of the angular velocities. The latter are actually moments caused by centrifugal forces acting on the limbs.

A final comment on the form of equation (3) is that the first three equations appear uncoupled from the last four, which is because the right foot, shank and thigh are separated from the left foot, shank, thigh and

HAT. The connection is at the hips with a spring-damper combination, but there is no direct coupling between the accelerations of the right limbs and those of the left limbs. Therefore, the coupling is through the b_i which include the spring forces caused by a small displacement of the two thighs at the hip joint.

A detailed description of equation (3) is given in the Appendix.

Next, consider the swing phase with the supporting foot pivoting either on the heel or the toe (Phase 2 and 4). The form of the corresponding equations of motion is,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & a_{55} & a_{56} & a_{57} \\ a_{61} & a_{62} & a_{63} & 0 & a_{65} & a_{66} & a_{67} \\ a_{71} & a_{72} & a_{73} & 0 & a_{75} & a_{76} & a_{77} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \\ \ddot{\theta}_6 \\ \ddot{\theta}_7 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} \tag{5}$$

or,

$$[a_{ij}]\ddot{\theta}_i = b_i.$$

As in the previous case, the a_{ij} are symmetrical and of the form shown in equation (4), however, they take on different values. The b_i are similar, but do not include terms involving the spring-damper combination because it does not now exist. Again, a detailed description of equation (5) is given in the Appendix.

Finally, during Phase 3 with the supporting foot constrained against the floor the resulting equations of motion are identical to those shown in equation (5), except the first row and first column are eliminated. This accounts for the fact that the dynamics of the supporting foot do not enter into the equations of motion.

IMPLEMENTATION OF THE MATHEMATICAL MODEL

The solution to the proposed problem requires the knowledge of the initial conditions, which are the angular displacements and velocities of the limbs, plus the system inputs, which are the joint moments. These data were obtained from the Gait Laboratory at the University of Waterloo, Ontario, where large files of data exist of locomotion runs of individuals. The initial angular conditions had to be adjusted slightly to make them consistent, such that when applied to the segment lengths, this results in having both feet on the floor. Also, during double support, the angular velocities must be such that the top of the right thigh moves at the same velocity as the top of the left thigh.

Again, due to differences between the mathematical model and the human body, the calculated joint moments were not exactly what is required for the link segment model to progress through a desirable walking cycle. Consequently, the joint moments were adjusted, as required, to attain a desirable gait. The adjustments were made "manually" by inspecting the results and using judgment based on previous runs. It should be emphasized that one does not necessarily have to use measured moment patterns. Instead, one may want to experiment with a variety of conditions. The anthropometric data required in the equations of motion equations (3) and (5) are given in the Appendix in Table A1.

The equations of motion are complex and nonlinear; therefore, the only practical approach is to use the digital computer for obtaining the solution, which is the set of limb angles as a function of time or of percent walking cycle. The approach used to finding the solution is as follows. Starting with the initial conditions one can find the a_{ij}, which depend on the angles, and the b_i which depend on these same angles and the angular velocities (see equations (3) and (5)). Having the values of a_{ij} and b_i at the initial time one can solve for the angular accelerations at this time by solving equation (3) or (5), depending on the walking phase. Next, the angular position and velocity can be found at some time interval later using any of a number of numerical integration techniques. In this case, Euler's scheme was used, i.e.

$$\theta_i(t+h) = \theta_i(t) + h\dot{\theta}_i(t), \tag{6}$$

where h is the integration interval. It was found that the set of equations in equations (3) and (5) were not ill-conditioned and this approach worked well. Perhaps, other more sophisticated techniques should have been used, however, this one is simple and it did the job.

Having the integration process established, the next consideration is the determination of the different phases of walking as the solution progresses. The solution begins in Phase 1 which ends at left-toe-off (LTO). To determine LTO, the vertical floor reaction force on the left toe is calculated every integration interval. Initially, this force pushes up on the left toe, and, as soon as it becomes zero or just changes sign Phase 1 of walking ends. During this phase both feet are pivoting on the floor and there is a spring-damper combination at the hip. The vertical reaction force on the left toe depends on the vertical accelerations of the four segments connected to the left toe (left foot, shank, thigh and HAT), the gravitational forces on these segments, and the vertical force of the spring-damper combination at the hip.

Phase 2 begins with the equations of motion corresponding to the beginning of the swing phase, as illustrated in equation (5). Phase 2 had the right foot pivoting on the heel but the floor reaction force is not calculated anymore, instead, the right foot angle is monitored. Throughout Phase 2 this angle is positive and at RFF it becomes zero, determining the end of Phase 2.

Phase 3 covers the single-support phase when the right foot is constrained against the floor, and ends when right-heel-off (RHO) occurs. The equations of motion are the same as those used in Phase 2, but with the first row and first column eliminated. To determine the end of Phase 3, the moment on the right foot acting about the toe is monitored. This moment is a function of the reaction forces at the ankle, the joint moment at the ankle, and the weight of the foot itself. The reaction forces at the ankle are found by accounting for the accelerations of, and gravitational forces acting on the six segments supported by the ankle. During Phase 3 the net moment on the foot about the toe acts in the direction to keep this foot against the floor. As soon as this moment changes sign to lift the heel, Phase 3 ends.

The final phase, Phase 4, uses the equations of motion shown in equation (5), with the right foot pivoting on the toe. The end of Phase 4, and, thus, the end of the step is determined by monitoring the height of the left heel during Phase 4. When this heel reaches the floor, the walking step has ended and the calculations cease.

RESULTS

The computer program that was developed includes the equations of motion and the required logic for progressing through the different phases of the walking cycle. Starting with a given set of initial kinematic conditions and time histories of the six joint moments the resulting kinematics of the various limbs were obtained. These were inspected, the joint moments were adjusted during each of the four phases of walking, and the program was run again. This procedure was repeated until the model progressed through a complete walking cycle. When the final conditions were within a few percent of the initial conditions, the calculations were terminated. Instead of presenting vast tables of numbers, the results of this procedure are given in stick figure form in Fig. 3. Position (a) shows the initial conditions, (b) shows right-foot-flat (13% of step cycle) and (c) shows right-heel-off (57% of step cycle). Positions (d) and (e) show the stick figure at 82 and 100% of the step cycle, respectively. One hundred percent corresponds to left-heel-strike. The time to complete the step is 0.55 sec. Left-toe-off, which is not shown in Fig. 3, occurs at 8% of the cycle. This is somewhat early; however, it is at least partly due to the fact that foot in the proposed model does not include a toe.

Another response of the proposed model is shown in Fig. 4. In this case the applied joint moments are identical except the left ankle moment is increased by 20% in Phase 1, which lasts until left-toe-off. Even though Phase 1 lasts for only 8% of the cycle, it has a significant effect on the response. Positions (a), (b), (c) and (d) show the limbs initially, at right-foot-flat, at right-heel-off and at left-heel-strike, respectively. Left-heel-strike occurs quite early, at 0.45 sec which corresponds to the point in time of position (d) in Fig. 3. Comparing the corresponding stick figures in Figs. 3 and 4, the increased left ankle "push-off" propels the body faster, resulting in a shorter time to complete the step and in a shorter step size. With this perturbation in the left ankle moment, the stick figure quickly "falls down".

A third response, not reported in stick figure form, involved a decrease of the left ankle moment during late push-off. This caused the body to not progress over the right leg properly. Eventually, the right leg buckled slightly and the step ended early and with a short step length.

Other intuitively obvious effects have been observed while exercising the developed model. Three of these, for example, are: (1) A higher dorsi-flexion moment after right heel contact at the right ankle delays foot-flat, (2) An increase in the initial left ankle moment results in an increase in the floor reaction force, and (3)

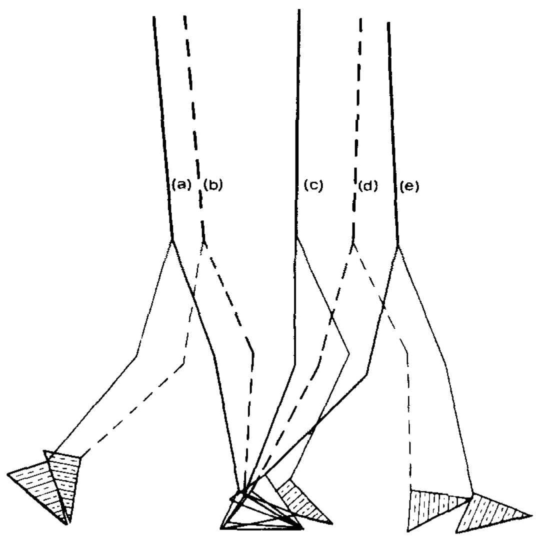

Fig. 3. Positions of limb segments, (a) initially, (b) at right-foot-flat, (c) at right-heel-off, (d) at 82% of the step cycle, (e) at end of step. For detailed discussion see text.

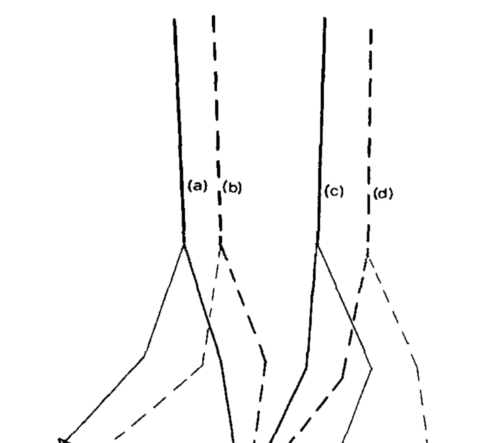

Fig. 4. Limb positions with left ankle moment increased during push-off, (a) initially, (b) at right-foot-flat, (c) at right-heel-off, (d) at end of step. For detailed discussion see text.

An incremental change in the knee moment, for example, can be compensated for by appropriate changes at the hip and ankle.

SUMMARY AND CONCLUSIONS

A mathematical model using Lagrangian mechanics was developed to simulate human gait. A specific set of initial conditions and adjusted joint moments were used to demonstrate that this model progresses through a walking step, as desired. With simple perturbations of the joint moments the model responds as one would intuitively predict. It is anticipated that this model will provide insight into how specific or complex changes in muscle activity will affect human gait. Thus, it is anticipated that it will be valuable as a teaching and diagnostic tool.

Future refinements to this model will include the natural ligamentous constraints that exist in human joints. For example, hyper-extension of the knee will be limited to a few degrees. Also, the foot will be refined to include the spring-like effect of the toes at push-off. Another helpful addition now being developed are interactive computer graphics to assist the operator in perturbing the model and seeing the results.

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APPENDIX

Included here are a list of anthropometric constants in Table A1 and details of the equations of motion describing human gait. There are two basically different mathematical representations to cover the different phases of walking, one for the double support phase, Phase 1, and one for the single support phase, Phases 2, 3 and 4. The equations of motion for the beginning and end of the single support phase, Phases 2 and 4, are identical except that in Phase 2 the right foot pivots on the heel, i.e. the foot link is represented by the heel to ankle distance. Whereas, in Phase 4 the right foot pivots on the toe, thus, the toe to ankle distance is used for the foot segment length. Phase 3, when the right foot is flat, involves a larger variation because the right foot is constrained against the floor; however, as it turns out, one needs only to set the first row and column in equation (5) to zero to account for this. Therefore, the swing phase is covered with essentially one set of equations of motion.

The form of the equations of motion during the double support, Phase 1, as given in equation (3) is,

$$[a_{ij}]\ddot{\theta}_i = b_i, \quad i = 1,\ldots,7;\ j = 1,\ldots,7, \tag{A1}$$

where the terms a_{ij} are symmetrical. Here the right foot is pivoting about the heel and the left foot about the toe, therefore, the required right foot and left foot angles are θ_1 +

Table A1. Anthropometric and related constants

Symbol	Parameter (*CG* denotes center of gravity)	Value (all MKS)
B	heel to ankle distance	0.140
C	heel to *CG* distance	0.189
A	ankle to *CG* distance	0.116
m_f	foot mass	1.159
I_f	foot moment of inertia about *CG*	0.010
α	angle between sole and ankle measured at the toe	2.621
β	angle between sole and ankle measured at the heel	0.967
γ	angle between sole and *CG* measured at the heel	0.310
D	ankle to shank *CG* distance	0.247
E	shank *CG* to knee distance	0.188
m_s	shank mass	3.717
I_s	shank moment of inertia about *CG*	0.064
F	knee to thigh *CG* distance	0.227
G	thigh *CG* to hip distance.	0.173
m_t	thigh mass	7.994
I_t	thigh moment of inertia about *CG*	0.133
H	hip to *CG* of head, arms and trunk (HAT) distance	0.325
m_h	HAT mass	54.2
I_h	HAT moment of inertia about *CG*	3.591
W_m	total body mass	79.9
S_k	spring constant at hip	274,400
f	damping coefficient of damper in hip	1,000
g	gravitational constant	9.807

Table A2. Constants c_{ij} for Phase 1

$$
\begin{aligned}
c_{11} &= m_f C^2 + I_f + (m_s + m_t)B^2 \\
c_{12} &= m_s DB + m_t(D + E)B \\
c_{13} &= m_t FB \\
c_{22} &= I_s + m_s D^2 + m_t(D + E)^2 \\
c_{23} &= m_t F(D + E) \\
c_{33} &= I_t + m_t F^2 \\
c_{44} &= I_h + m_h H^2 \\
c_{45} &= m_h H(F + G) \\
c_{46} &= m_h H(D + E) \\
c_{47} &= 2m_h HA \\
c_{55} &= I_t + m_t F^2 + m_h(F + G)^2 \\
c_{56} &= m_t F(D + E) + m_h(D + E)(F + G) \\
c_{57} &= 2m_t FA + 2m_h A(F + G) \\
c_{66} &= I_s + m_s D^2 + (m_h + m_t)(D + E)^2 \\
c_{67} &= 2(m_h + m_t)(D + E)A + 2m_s AD \\
c_{77} &= I_f + (0.25m_f + m_h + m_t + m_s)4A^2.
\end{aligned}
$$

β and $\theta_7 + \alpha$, respectively. To simplify the expressions for a_{ij} and b_i given below, it will be assumed that during Phase 1, θ_1 and θ_7 include the β and α, respectively. The expression for a_{ij} is,

$$a_{ij} = c_{ij}\cos(\theta_i - \theta_j), \quad i = 1,\ldots,7;\ j = i,\ldots,7. \quad \text{(A2)}$$

The expressions for the c_{ij} for Phase 1 are given in Table A2. It should be pointed out that the required variables and constants needed in c_{ij} and in b_i are defined in Figs. 1, 2 and in Table A1. Also, the c_{ij} not given in the Table are zero and all are symmetrical (see the zeros in the left hand side of equation (3)).

Before providing the expressions for the b_i it is convenient to define the following set of constants, d_j,

$$
\begin{aligned}
d_1 &= B \\
d_2 &= D + E \\
d_3 &= F + G \\
d_4 &= 0 \\
d_5 &= -(F + G) \\
d_6 &= -(D + E) \\
d_7 &= -2A.
\end{aligned}
\quad \text{(A3)}
$$

The b_i are given by,

$$b_i = -\sum_{j=1}^{7} c_{ij}\dot{\theta}_j^2 \sin(\theta_i - \theta_j) + S_k\left[\sum_{j=1}^{7} d_i d_j \sin(\theta_i - \theta_j) + d_i S \sin\theta_i\right] + R_i, \quad \text{(A4)}$$

where

S = initial left-toe to right-heel distance

$$
\begin{aligned}
R_1 &= -m_f gC\cos(\theta_1 - \beta + \gamma) - (m_s + m_t)gB\cos\theta_1 + M_{ra} + D_1 \\
R_2 &= -C_{21}g\cos\theta_2 + M_{rk} - M_{ra} + D_2 \\
R_3 &= -C_{31}g\cos\theta_3 + M_{rh} - M_{rk} + D_3 \\
R_4 &= -m_h Hg\cos\theta_4 - M_{rh} - M_{lh} + D_4 \\
R_5 &= -[m_h(F + G) + m_t F]g\cos\theta_5 + M_{lh} - M_{lk} + D_5 \\
R_6 &= -[(m_h + m_t)(D + E) + M_s D]g\cos\theta_6 + M_{lk} - M_{la} + D_6 \\
R_7 &= -[2A(m_h + m_t + m_s) + Am_f]g\cos\theta_7 + M_{la} + D_7,
\end{aligned}
\quad \text{(A5)}
$$

where the D_i are moments caused by the damper in the hip. To determine these moments, one determines the virtual work done by the damper force. When this virtual work expression is in the following form, the coefficients of the $\delta\theta_i$ become the desired moments for the θ_i equation of motion.

$$\delta W_D = D_1\delta\theta_1 + D_2\delta\theta_2 + \ldots + D_7\delta\theta_7. \quad \text{(A6)}$$

One way to find the virtual work is to begin in the X-Y coordinate system. The damper force is proportional to the relative velocity of the tops of the right and left thighs. Writing the expressions relative to the left thigh, the damping force F_D is,

$$F_D = -f(v_R - v_L), \quad \text{(A7)}$$

where v_R and v_L are the velocities of the top of the right and left thighs, respectively. Writing equation (A7) in component form,

$$
\begin{aligned}
F_{DX} &= -fv_x \\
F_{DY} &= -fv_y,
\end{aligned}
\quad \text{(A8)}
$$

where v_x and v_y are the X and Y components of the desired relative velocity. Note that F_{DX} and F_{DY} are not a function of virtual displacements and can easily be found by resolving the individual segment velocities into X and Y components.

Table A3. Constants c_{ij} for Phases 2, 3 and 4

$$
\begin{aligned}
c_{11} &= K^2(2m_s + 2m_t + m_h + m_f) + m_f Z^2 + I_f \\
c_{12} &= K[m_s D + (2m_t + m_h + m_s + m_f)(D + E)] \\
c_{13} &= K[m_t + F + (m_h + m_t + m_s + m_f)(F + G)] \\
c_{14} &= Km_h H \\
c_{15} &= -K[m_t G + (m_s + m_f)(F + G)] \\
c_{16} &= -K[m_s E + m_f(D + E)] \\
c_{17} &= -Km_f A.
\end{aligned}
$$

In above, $K = B$ in Phase 2, zero in Phase 3 and $2A$ in Phase 4; $Z = C$ in Phase 2, zero in Phase 3 and A in Phase 4. Below, applies to all three Phases.

$$
\begin{aligned}
c_{22} &= I_s + m_s D^2 + (m_s + 2m_t + m_h + m_f)(D + E)^2 \\
c_{23} &= [m_t F + (m_h + m_t + m_s + m_f)(F + G)](D + E) \\
c_{24} &= m_h H(D + E) \\
c_{25} &= -[m_t G + (m_s + m_f)(F + G)](D + E) \\
c_{26} &= -[m_s E + m_f(D + E)](D + E) \\
c_{27} &= -m_f A(D + E) \\
c_{33} &= I_t + m_t F^2 + (m_h + m_t + m_s + m_f)(F + G)^2 \\
c_{34} &= m_h H(F + G) \\
c_{35} &= -[m_t G + (m_s + m_f)(F + G)](F + G) \\
c_{36} &= -[m_s E + m_f(D + E)](F + G) \\
c_{37} &= -m_f A(F + G) \\
c_{44} &= I_h + m_h H^2 \\
c_{45} &= c_{46} = c_{47} = 0 \\
c_{55} &= I_t + m_t G^2 + (m_s + m_f)(F + G)^2 \\
c_{56} &= [m_s E + m_f(D + E)](F + G) \\
c_{57} &= m_f A(F + G) \\
c_{66} &= I_s + m_s E^2 + m_f(D + E)^2 \\
c_{67} &= m_f A(D + E) \\
c_{77} &= I_f + m_f A^2.
\end{aligned}
$$

Now, the virtual work done by the damper, expressed in the X-Y coordinates is,

$$\delta W_D = F_{DX}\delta x + F_{DY}\delta y. \qquad (A9)$$

Expressing δx and δy in terms of the angular displacements, writing δW_D in the form shown in equation (A6), the required moments, D_i, caused by the damper are,

$$D_i = d_i(-F_{DX}\sin\theta_i + F_{DY}\cos\theta_i). \qquad (A10)$$

This completes the detailed description of the equations of motion for Phase 1. Before proceeding, it may be of interest to note the make-up of b_i in equation (A4). The first summation in the right hand side involves moments due to centrifugal force terms and the second summation due to the spring in the hip. The remaining term R_i includes moments due to gravitational forces, the applied joint moments and the effect of the damper in the hip.

Next, the equations of motion for the swing phase, Phases 2, 3 and 4, will be presented together. These equations of motion are similar in form to those for Phase 1 as shown in equation (A1), except the coefficients c_{ij} are different, and, because there is no spring-damper combination at the hip the second summation and the D_i terms in the right hand side of b_i in equation (A4) do not exist. Therefore, the expression for a_{ij} for Phases 2, 3 and 4 is the same form as given in equation (A2), except for the c_{ij} which are given in Table A3. Again, the c_{ij} are symmetrical. The right hand side of equation (A1) is given by,

$$b_i = -\sum_{j=1}^{7} c_{ij}\dot{\theta}_j^2\sin(\theta_i - \theta_j) + R_i, \qquad (A11)$$

where

$$R_1 = -Bgm_f\cos(\theta_1 - \beta + \gamma) - (2m_s + 2m_t + m_h + m_f)\cos\theta_1 + M_{ra}, \quad \text{Phase 2}$$

$$R_1 = 0, \quad \text{Phase 3}$$

$$R_1 = -2Agm_f\cos(\theta_1 - \beta + \gamma) - (2m_s + 2m_t + m_h + m_f)\cos\theta_1 + M_{ra}, \quad \text{Phase 4}$$

$$R_2 = -c_{12}\cos\theta_2 + M_{rk} - M_{ra}, \quad \text{all phases} \qquad (A12)$$

$$R_3 = -c_{13}g\cos\theta_3 + M_{rh} - M_{rk}, \quad \text{all phases}$$

$$R_4 = -c_{14}g\cos\theta_4 - M_{rh} - M_{lh}, \quad \text{all phases}$$

$$R_5 = -c_{15}g\cos\theta_5 + M_{lh} - M_{lk}, \quad \text{all phases}$$

$$R_6 = -c_{16}g\cos\theta_6 + M_{lk} - M_{la}, \quad \text{all phases}$$

$$R_7 = -c_{17}g\cos\theta_7 + M_{la}, \quad \text{all phases.}$$

This completes the detailed description of the equations of motion.